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On The Lyapunov Theory: Application To LTI Systems

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Abstract

This brief paper is a review of the classical Lyapunov theory, also referred to as Lyapunov's direct method or Lyapunov's second method. The Lyapunov theory uses the concept of Lyapunov function of the system to draw conclusions on system stability, without actually solving the differential equation of the system motion. Some primary definitions are recalled and uniqueness solution issue for the Lyapunov equation is addressed for linear systems.

Key words: *Asymptotic Stability, Lyapunov Stability, Lyapunov function, Linear Time Invariant System (LTI).*

1. Introduction

In Control theory, stability is the basic requirement for prior to any attempt to make use of any system. One method of investigating stability, widely used, is the Lyapunov theory introduced by Alexandr Mikhailovich Lyapunov (1892). In his approach, the energy function of the system, or any other similar function (later on named after him: a Lyapunov function) guarantees stability of the system. In this review, it is recalled that for LTI systems, the Lyapunov function is merely a quadratic form of the state vector and the overall process of investigating stability ends up in solving the Lyapunov equitation. The solution,
when it exists, is unique.

2. Preliminaries

2.1. Abbreviations And Symbols

$\mathbb{R}$ Set of real numbers

$\mathbb{R}^n$ n dimensional euclidian space

$x(t)$ State vector $x(t) \in \mathbb{R}^n$

$\|x\|$ Euclidian norm $= \sqrt{x^T x}$ of $x(t)$

$O_\delta(x_0)$ Neighbourhood of x_0 of radius $\delta \in \mathbb{R}$

$V = V(x,t)$ Scalar Function $V: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$

$\dot{V}(x,t)$ Time dérivative of V ; $\frac{dV}{dt} = \nabla V . f(x)$

∇V Gradient of V with respect to $x(t)$; $\nabla V = \frac{\partial V}{\partial x} = [\frac{\partial V}{\partial x_1}, \quad \cdots \quad \frac{\partial V}{\partial x_2}]^T$

LTI Linear Time Invariant System

$P = P^T > O$ P symmetric positive definite matrix as in $x^T P x > 0$ if $x \neq 0$.

2.2. Basic Definitions

Let $x(t) \in \mathbb{R}^n$; this is referred to as state vector of an autonomous system in (1).

$\dot{x} = f(x) \qquad \ldots \qquad (1)$

Definition 1: Equilibrium Point

If $x_0 = x\,(t_0)$; $f(x_0) = 0$, $t_0 \in \mathbb{R}$, then x_0 is **an equilibrium point**.

$O_d(x_0) \equiv \{x: \|x - x_0\| < d\}$; $\|x\| = \sqrt{x^T x}$ is the Euclidian norm of $x(t)$.

Definition 2: Stable Equilibrium Point

If x_0 is an equilibrium point , then $x(t) = x_0$ is a trajectory of the system.

An equilibrium point x_0 is **stable** in the sense of Lyapunov if :

$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 : \quad x(t_0) \in O_\delta(x_0) \Rightarrow x(t) \in O_\varepsilon(x_0) \quad , \forall\, t > t_0$.

Any trajectory starting close to the equilibrium point remains close to it.

Definition 3: Asymptotic Stability / Asymptotic Stability in the large

An equilibrium point $x_0 = x(t_0)$ is asymptotically stable in region D if :

$x(t_0) \in D \Rightarrow x(t) \rightarrow x_0$ as $t \rightarrow +\infty$.

Any trajectory starting sufficiently close to the equilibrium point, will eventually approach it.This ius depicted in figure Figure 1.(a).

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An equilibrium point is said to be asymptotically stable in the large if it is asymptotically stable and every motion starting at any point in the state space , returns to that point as t tends to infinity as shown in figure Figure 1.(b).

It is obvious from the above that asymptotic stability is equivalent to Lyapunov stability. Lyapunov theory is used to make conclusions about trajectories of system (1) without finding the trajectories, i.e., one does not have to solve the differential equation.

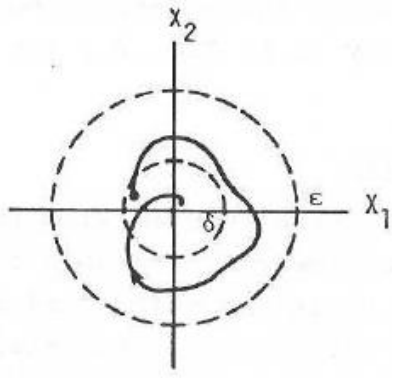

Figure 1.(a) Asymptotic Stability

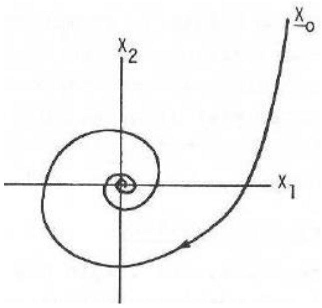

Figure 1.(b) Asymptotic Stability in the large

<u>Definition 4:</u> Positive Definite Function

A scalar function $V = V(x,t)$ from $x(t) \in \mathbb{R}^n$ to $\mathbb{R}$, is said to be **positive definite** if

i) $V = 0 \Leftrightarrow x = 0;$

ii) $V \neq 0$ if $x \neq 0$

iii) $V > 0$ if $x \neq 0$

<u>Definition 5</u> : Lyapunov Function

Any scalar positive definite function $V = V(x,t)$ that satisfies condition state in (1), is called a **Lyapunov function** for the system.

$$\dot{V}(x,t) = \frac{dV}{dt} < 0 \quad \ldots \quad (1)$$

<u>Definition 6</u> : Lie Derivative

Given s scalar function $V = V(x,t)$, the time derivative can be computed as in equation (2).

$$\dot{V}(x,t) = \frac{dV}{dt} = \nabla V . f(x) \ldots\ldots\ldots \; (2)$$

The vector $\nabla V\,(x,t)$ is the gradient of the scalar function $V = V(x,t)$ and the dot product $\nabla V . f(x)$ of the gradient vector of $V = V(x,t)$ with the vector field $f(x)$ in equation (2) , is referred to as directional derivative of $V = V(x,t)$ along the vector field $f(x)$ or the **Lie derivative** of $V = V(x,t)$ along $f(x)$.

2.3. Lyapunov's Second Method (Direct Method)

The Lyapunov's Second Method, which is now referred to as the Lyapunov Stability Theorem, makes use of a Lyapunov function to check the stability of an equilibrium point of a system.

The system (1) is stable if there exists a Lyapunov function $V = V(x,t)$ for it.

2.3.1. Lyapunov Criterion And Lyapunov Equation

An autonomous linear system is given in equation (3).

$$\dot{x} = Ax \quad \ldots \quad (3)$$

The matrix A is Hurwitz (its eigenvalues have negative real parts). The existence of a Lyapunov function $V(x)$ as in equation (4), where $P = P^T > O$, a positive definite symmetric matrix, guarantees stability of the system in (3). In other words, this is a sufficient condition for stability.

$$V(x) = x^T P x \;\ldots \quad (4)$$

$V(x) > 0$ for all $x\,(t)$ and $V(x) = 0$ if and only if $x(t) = 0$.

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = (Ax)^T P x + x^T P A x = x^T(A^T P + PA)x \leq O$$

for all $x \neq 0$.

There exists $Q = Q^T > 0$ so that $\dot{V}(x) = \frac{dV}{dt} = x^T(A^T P + PA)x = -x^T Q x$

This provides necessary and sufficient conditions for stability of the autonomous system (3).

This can be expressed as in (5) and (6) below :

$$A^T P + PA \leq O \quad \ldots \quad (5)$$

$$A^T P + PA = -Q \quad \ldots \quad (6)$$

The expressions (5) and (6) are the Lyapunov inequation and equation, respectively.

The matrix $P = P^T > 0$ is to be found while $Q = Q^T > 0$ is given. The matrix A must be Hurwitz ; in other words, all eigenvalues must have negative real parts.

$P^T = P > O$, then the eigenvalues are real.

$\|P\| = \lambda_{max}(P) =$ the maximum eigenvalue of P. $R(x) = \frac{x^T P x}{x^T x} = \frac{x^T P x}{\|x\|^2}$

Any vector x that minimizes $R(x)$ is an eigenvector $Ax = \lambda x$.

And so $\lambda = R(x) = \frac{x^T P x}{x^T x} =$ for that eigenvector $R(x) = (\frac{x}{\|x\|})^T P \left(\frac{x}{\|x\|}\right) = v^T P v\,;\quad v = \frac{x}{\|x\|}$

$$\lambda_{min}(P) \leq R(x) \leq \lambda_{max}(P)$$

$$min\, R(x) = \lambda_{min}(P)$$

$$max\, R(x) = \lambda_{max}(P)$$

$$0 \leq \lambda_{min}(P)\, \|x(t)\|^2 \leq x^T(t) P x(t) \leq \lambda_{max}(P)\, \|x(t)\|^2$$

$$0 \leq \lambda_{min}(P)\, x^T x \leq x^T P x \leq \lambda_{max}(P)\, x^T x$$

$$\dot{V}(x) = -x^T Q x \leq -\lambda_{min}(Q) x^T x \leq \lambda_{min}(Q)$$

$$\leq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} x^T P x = -\alpha V(x) \leq 0$$

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We have proved that , in liear case, a quadratic Lyapunov function is deacreasing with time (Raktim Bhattacharya, 2023).

Given a particular $Q > O$, the P matrix is found as in equation (7).

$$P = \int_0^{\infty} e^{A^T t} Q e^{At}\, dt \quad \ldots \quad (7)$$

Proof:

$$A^T P + PA$$

$$= \int_0^{\infty} (A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A) dt = \int_0^{\infty} \frac{d}{dt}(e^{A^T t} Q e^{At}) dt$$

$$= e^{A^T t} Q e^{At} \Big|_0^{\infty} = -Q.$$

2.3.2. Uniqueness of Solution to the Lyapunov Equation

Assume now that P satisfies the matrix Lyapunov equation (7). Suppose that there exists another matrix, say $\tilde{P}$ that also solves (6).

Then we have equations (6) or (8) and (9) that simultaneously hold.

$$A^T P + PA = -Q \quad \ldots \quad (8)$$

$$A^T \tilde{P} + \tilde{P} A = -Q. \quad \ldots \quad (9)$$

Subtracting (9) from (8) we get (10).

$$A^T (P - \tilde{P}) + (P - \tilde{P}) A = O \quad \ldots \quad (10)$$

This implies that $e^{A^T t}[A^T (P - \tilde{P}) + (P - \tilde{P}) A] e^{At} = \frac{d}{dt}[e^{A^T t}(P - \tilde{P}) e^{At}] = O$

Therefore $e^{A^T t}(P - \tilde{P}) e^{At}$ = constant matrix M ; for any value of t.

In particular, $M = P - \tilde{P}$ for $t = 0$ and if t goes to infinity, then $M = O$.

This means that $P - \tilde{P} = O$ *or* $P = \tilde{P}$.

In other words, the solution, when exists, is unique!

2.3.3. Application

The following in (14) is a Lyapunov Function to the system (11).

$$V(x) = x^T \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} x \quad \ldots \quad (14)$$

It has applied the concept to a nonlinear damped pendulum. Phase portraits sketches have been shown for some arbitrary initial conditions. The intent was to keep the paper short , so as to keep attention to the main concern : stability in the sense of Lyapunov.

Example:

$$A = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$$

In MATLAB, type : **help lyap**

Solve continuous-time Lyapunov equations.

X = lyap (A, X) solves the matrix Lyapunov equation **AX + XA' + Q = 0**

Be careful for the syntax!

PA + A'P + P = 0

To solve for matrix P above, we should write : **lyap (A', P)**

It appears that the MATLAB Lyapunov equation is the transpose of the Lyapunov equation we actually use! Assuming **Q = I** ; then **lyap (A', I)**

$$\boldsymbol{P} = \begin{pmatrix} \mathbf{1.3333} & \mathbf{0.16671} \\ \mathbf{0.1667} & \mathbf{0.3333} \end{pmatrix}$$

3. Conclusion

This paper has presented the Lyapunov stability theory and its application to an LTI. The keystone principle is the concept of Lyapunov function. In linear systems, the whole process consist of solving the so called Lyapunov equation who has a unique solution, when exists.

Appendix A1

Matrix Exponential $\phi(t) = e^{At}$

This matrix $\phi(t) = e^{At}$ helps to solve the equation A1.1.

$$\dot{x} = \frac{dx}{dt} = Ax . \quad \ldots \text{ A1.1.}$$

Where $x(t) \in \mathbb{R}^n$, with $x(0) = x_0$ A is an $n \times n$ matrix.

In the literature, there are many ways of getting $\phi(t)$. One way is through the Laplace Transform. The bounded causal signal $y(t)$ has the Laplace Transform $Y(s)$ defined in A1.2.

$$Y(s) = \int_0^{\infty} y(t) e^{-st}\, dt \quad \ldots \text{ A1.2.}$$

Taking the transform for each side in equation A1.1; one gets the following equation $sX(s) - x_0 = A\ X(s)$. or finally as equation A1.3.

$$X(s) = (sI - A)^{-1} x_0. \ldots \text{ A1.3.}$$

Equation A1.3 is the Laplace Transform of equation A1.4 , in time domain.

$$x(t) = \phi(t) x_0 = e^{At} x_0. \quad \ldots \text{ A1.4.}$$

Hence $(sI - A)^{-1}$ is Laplace Transform of $\phi(t) = e^{At}$.

The following properties hold for the matrix exponential $\phi(t) = e^{At}$.

a) $\phi(t) = e^{At}$.

b) $\phi(t) = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

c) $\phi(0) = I$

d) $\frac{d}{dt}(\phi(t)) = A\phi(t)$

e) $(\phi(t))^{-1} = \phi(-t) = e^{-At}$

f) $B\phi(t) = \phi(t)B$ if $BA = AB$

Appendix A2

A2.1. Symmetric Positive Definite Matrix

The primary definition of a positive definite matrix P is the following.

For any vector $x = x(t) \in \mathbb{R}^n$, if $x(t) \neq 0$; we have $x^T P\, x > 0$. This is written as $P > 0$. This definition, based on the energy (Lyapunov's function) is fundamental in control systems.

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It can be shown that such a matrix P is necessarily symmetric (because only a real symmetric matrix has its eigenvalues real) ! Hence, the notation $P = P^T > O$

Given two positive definite matrices $P = P^T > O$, $Q = Q^T > O > 0$;

It is also true that $(P+Q) > 0$ and $\begin{pmatrix} P & O \\ O & Q \end{pmatrix} > 0$.

If $P > O$ (positive definite) , then $-P < O$ (negative definite).

A matrix that is neither positive definite nor negative definite, is indefinite.

A2.2. Equivalent Statements for a Definite Matrix

To begin with, the matrix P is a positive definite matrix if and only if P is symmetric and $P^T = P > O$. For any given positive definite square matrix $P > O$ of size n, the statements (i), through (x) are equivalent.

i. The n pivots of P are strictly positive (they are reals).

ii. The n determinants of the main diagonal of P are positive.

iii. The n eigenvalues of P are strictly positive (they are reals).

iv. For any vector $= x(t) \in \mathbb{R}^n$, if $x(t) \neq 0$; we have $x^T P\, x > 0$.

v. $P = R^T R$ where R has its columns linearly independent.

vi. The Cholesky's decomposition $P = LL^T$ is possible.

vii. The n determinants referred to in above, are defined as $|P_1| = [p_{11}]$; $|P_2| = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$;…; $|P_n| = \det(A)$. This the Sylvester criterion.

viii. The Schur complement is positive

ix. If $P > 0$ then $P^{-1} > 0$ and hence P is non-singular.

x. $P = U\Lambda U^T$; $UU^T = I$.

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