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CRITICAL REVIEW OF FUNDAMENTAL CONCEPTS IN PHYSICS Part 6 – Critical Notes on the Problem of Elastic Scattering of Particles in Non-Relativistic Quantum Mechanics

By

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Abstract

This article critically examines the details of the corresponding theoretical problem of particle scattering in non-relativistic quantum mechanics. The discussion proceeds in two directions: one - the correctness of physical and mathematical interpretations of individual details of the scattering problem; second - checking the correctness of mathematical methods used in specific mathematical calculations. Details of the scattering problem are pointed out, whose existing interpretations contradict the basic principles of quantum mechanics. A detail of mathematical calculations is also pointed out, contradicting the principle of correct calculation and leads the corresponding calculations to erroneous results. After correcting this erroneous mathematical method, it becomes clear that the problem lies in the initial mathematical algorithm itself, which is used to formulate the theoretical problem of scattering.

Index Terms*- quantum superposition; Schrödinger stationary equation; scattering problem; mathematical algorithm; Born iteration method; Mott's formula.*

INTRODUCTION

The consideration of the scattering problem corresponding to elastic collisions played a significant role in the formation of quantum mechanical concepts. One such important issue is the phenomenon of interference of probability amplitudes. Quantum mechanics textbooks indicate two initial proofs of the quantum superposition phenomenon (see, e.g., (Feynman et al, 1963), (Feynman et al, 1965)):

1. "Physical alternatives, which cannot be distinguished experimentally from each other and whose corresponding probability amplitudes participate in quantum superposition sums, always interfere."
2. "Quantum superposition and the corresponding interference are purely quantum phenomena and have no analogues in classical mechanics."

These statements are considered one of the main characteristics of the quantum nature of the micro-world, and to indicate their essence, the authors of the textbooks (Feynman et al, 1963) and (Feynman et al, 1965) use the following examples: Let's say particles A and B collide along the X-axis and scatter by Coulomb interaction. The description is made in their center of mass system. Two detectors are mounted along the Y-axis from the center of the reference system in opposite directions, which we will mark with indices 1 and 2. Let $\Phi_{AB}(1;2)$ denote the state vector corresponding to the probability amplitude if after the collision - particle A is fixed in detector 1, and particle B in detector 2. Let $\Phi_{AB}(2;1)$ correspond to the probability amplitude when particle A is fixed in detector 2, particle B in detector 1. According to quantum mechanical concepts - when particles A and B are different, we can empirically distinguish the physical states corresponding to $\Phi_{AB}(1;2)$ and $\Phi_{AB}(2;1)$ from each other, and according to (Feynman et al, 1965) - only for this reason, these physical states represent mutually exclusive alternatives. In this case - the total probability that particles A and B will hit the first or second detector should be calculated by the standard rule of adding probabilities:

$$W(A;B) = |\Phi_{AB}(1;2)|^2 + |\Phi_{AB}(2;1)|^2 = 2\,P_{AB}\ ;$$

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This addition rule is standard in the sense that the same rule is used when describing probabilistic results of random events involving macro bodies. When particles A and B are identical, it becomes impossible to distinguish the physical states corresponding to the state vectors $\Phi_{AA}(1;2)$ and $\Phi_{AA}(2;1)$, and therefore - according to the same authors, the corresponding physical states should no longer be considered as mutually exclusive alternatives and in this case, to calculate the total probability we should use the quantum mechanical rule of adding probability amplitudes:

$W(A;A) = |\Phi_{AA}(1;2) + \Phi_{AA}(2;1)|^2$;

Since the state vectors corresponding to $\Phi_{AA}(1;2)$ and $\Phi_{AA}(2;1)$ are defined by the same Schrödinger equation, then even without solving the equation explicitly, we can assume that the conditions are satisfied:$\Phi_{AA}(1;2) = \Phi_{AA}(2;1)$. Accordingly - we will get:

$W(A;A) = |2\Phi_{AA}(1;2)|^2 = |2\Phi_{AA}(2;1)|^2 = 4P_{AA}$;

That is, in the case of right-angle scattering of identical particles, the numerical coefficient of the total probability turns out to be twice as large as in the case of different particles. This difference is considered a quantitative demonstration of the essence of the quantum interference phenomenon corresponding to quantum superposition.

Explicit indication of these pairs of probability amplitudes implies solving the corresponding Schrödinger equation. The explicit forms of these solutions should also confirm the fact of the quantitative difference between numerical coefficients of W(A; B) and W(A; A) mentioned above. In the case of identical particles, the corresponding explicit expression - on the example of elastic scattering of α-particles, was first indicated by Mott (see (Mott, 1930)). The obtained theoretical result was in full agreement with the above-mentioned rule of adding probability amplitudes. In the quantum mechanics literature, there is also a publication (see (Chadwick, 1930)) which describes the empirical picture of scattering formed as a result of α-particle collisions. The author of the publication - J. Chadwick, makes the claim: The results of the theoretical calculation obtained by Mott in the α-particle scattering problem are in agreement with the statistical picture obtained from the experiment.

This coincidence would indeed be an infallible argument for proving the real existence of the physical phenomenon corresponding to the interference mixing of probability amplitudes in quantum superposition sums, were it not for the question that we posed regarding this issue in Section 5 of the text (see (Baghaturia et al, 2025b)) and which also became the basis of the research presented in this publication: How correct were the mathematical calculations by which the theoretical expressions of the pair - $\{\Phi_{\alpha\alpha}(1;2);\ \Phi_{\alpha\alpha}(2;1)\}$ were obtained for the α-particles.

Below, we will analyze the details related to this issue, both from the point of view of mathematical correctness and the self-consistency of physical interpretations.

CHAPTER I: Brief History of the Issue and |in> States of the Scattering Problem

The formulation of the mathematical algorithm for the scattering problem begins with the analysis of the general form of the Schrödinger equation. In our case of interest - for α-particles, the Schrödinger equation has the form:

$$i\frac{\partial\Psi(\vec{R}_1;\vec{R}_2;t)}{\partial t} = \hat{H}(\vec{R}_1;\vec{R}_2)\,\Psi(\vec{R}_1;\vec{R}_2;t)\ ;$$

$$\hat{H}(\vec{R}_1;\vec{R}_2) = \hat{H}_0(\vec{R}_1) + \hat{H}_0(\vec{R}_2) + e^2/|\vec{R}_1 - \vec{R}_2|\ ; \qquad (1)$$

$\vec{R}_1$ and $\vec{R}_2$ are the radius vectors of the particles; $\hat{H}_0(\vec{R}_1)$ and $\hat{H}_0(\vec{R}_2)$ are the corresponding free Hamiltonian operators of the particles. Since the operators $\partial/\partial t$ and $\hat{H}$ commute, the eigenfunction of the Hamiltonian operator:

$$\hat{H}(\vec{R}_1;\vec{R}_2)\,\Psi(\vec{R}_1;\vec{R}_2) = E\,\Psi(\vec{R}_1;\vec{R}_2)\ ; \qquad (2)$$

should also be a solution of (1). Equation (2) is called stationary because it does not contain dependence on the time parameter. The operators $\vec{R}_1$ and $\vec{R}_2$ do not depend on the time parameter and in the coordinate representation, they correspond to multiplication by ordinary numbers. Note that solving the Schrödinger equation does not imply indicating the time dependence of coordinates or other physical characteristics. Therefore, the spectrum of eigenvalues - corresponding to individual physical characteristics, corresponds to sets of time-independent numbers that are defined as statistical sets as a result of empirical observation. Since the elements of statistical sets are not connected by dynamic links, only the numerical values of the state vector, for which the time equation (1) is written, are connected by chronological links. Since these solutions of the Schrödinger equation contain the eigenvalues of the $\vec{R}_1$ and $\vec{R}_2$ operators parametrically, this scheme of consideration implies the existence of a reference frame, in which - these quantities are considered as ordinary numbers. It is also implied that the reference system is inertial and therefore we can indicate quantitative physical regularities using them. It is also implied that the coordinates of the origin of the reference system correspond to zero numerical values and at the same time - the origin is stationary. Accordingly, it is implied that we can simultaneously and accurately indicate the numerical values of the coordinates and momenta of the origin of the reference system. In the quantum-mechanical description, such a quantum object does not exist, which indicates that the reference system of these descriptions corresponds to a classical object, i.e. - a macro-sized observer. Within the framework of such an interpretation it can be explained why numbers $\vec{R}_1$ and $\vec{R}_2$ are independent of time and their corresponding numerical values represent only a set of statistical numbers. In the Schrödinger equation, the status of all the quantum-mechanical operators eigenvalues can only be statistical and not chronological-dynamic. This fact is directly related to the observer factor introduced by Bohr in the quantum-mechanical description. The Ψ-functions of the solution of equations (1) and (2) should correspond to the same status: The changes in the mentioned circumstances will

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automatically lead to changes in these numerical distributions. If those changes occurs with some temporal regularity, this regularity should be reflected in the full Hamiltonian, by means of which - those changes will be reflected in the Ψ-functions as well. By means of the superposition sums constructed with the obtained spectrum of Ψ-functions, we will obtain the mathematical representations of quantum ensembles depending on the time parameter.

For reference systems of such nature, it is entirely permissible to introduce the center of mass system, which is easily implemented by the transformation of variables, defined in classical mechanics. As a result, (2) takes the form:

$$\hat{H}(\vec{R}\,;\vec{r})\,\Psi(\vec{R}\,;\vec{r}) = [\hat{H}_R(\vec{R}) + \hat{H}_r(\vec{r})]\,\Psi(\vec{R}\,;\vec{r}) = E\,\Psi(\vec{R}\,;\vec{r});$$

$$\hat{H}_r(\vec{r}) = -\,(M/2m_1m_2)(\frac{\partial}{\partial\vec{r}})^2 + e^2/|\vec{r}|;$$

$$\hat{H}_R(\vec{R}) = -\left(\frac{1}{2M}\right)(\frac{\partial}{\partial\vec{R}})^2\,; \qquad (3)$$

(In our case of interest - with identical particles - $m_1 = m_2$= M/2). Because the operators $\hat{H}(\vec{R}\,;\vec{r})$, $\hat{H}_R(\vec{R})$ and $\hat{H}_r(\vec{r})$ commute with each other, we can represent the function $\Psi(\vec{R}\,;\vec{r})$ in the form $\Psi(\vec{R};\vec{r}) = \Psi_R(\vec{R})\,\Psi_r(\vec{r})$, where:

$$\hat{H}_R(\vec{R})\,\Psi_R(\vec{R}) = E_R\,\Psi_R(\vec{R}); \quad \hat{H}_r(\vec{r})\,\Psi_r(\vec{r}) = E_r\,\Psi_r(\vec{r})\,;$$
$$E_R + E_r = E\,; \qquad (4)$$

If we take into account that the Hamiltonian of the center of mass $\hat{H}_R(\vec{R})$ commutes with the operator of the momentum of the center of mass, then the function $\Psi_R(\vec{R})$ can be represented as an eigenfunction of this momentum operator:

$$\Psi_R(\vec{R}) = \exp(i\vec{P}\vec{R})\,; \quad \vec{P}^2 = E_R; \qquad (5)$$

Unlike the center of mass Hamiltonian, the Hamiltonian corresponding to relative motion - $\hat{H}_r(\vec{r})$, does not commute with its corresponding momentum operator, which is why - the function $\Psi_r(\vec{r})$ cannot be represented in a form similar to (5). We can represent it in the form of a superposition of plane waves, which at the same time corresponds to the Fourier transform operation:

$$\Psi_r(\vec{r}) = \int d\vec{q}\,\exp(i\vec{q}\vec{r})\,\Psi_r(\vec{q})\,; \qquad (6)$$

Such a $\Psi_r(\vec{r})$ function no longer represents an eigenfunction of the momentum operator corresponding to the $\vec{r}$ -variable with a fixed eigenvalue and can be used as a form to search for eigenfunctions of $\hat{H}_R(\vec{R})$.

Let's analyze the mathematical algorithm for determining $\Psi_r(\vec{r})$ indicated in quantum mechanics textbooks (see, for example, (Schweber, 1961), (Davydov, 1976), (Blokhintsev, 1964)). In the stationary equation:

$$[\frac{\partial}{\partial\vec{r}}^2 + k^2\,]\,\Psi_r(\vec{r}) = (g/|\vec{r}|)\,\Psi_r(\vec{r});$$
$$k^2 = mE_r; \quad g = me^2\,; \qquad (7)$$

The authors introduce the so-called "asymptotic |in> state" in the solution, for which $\Psi_r(\vec{r})$ is represented as:

$$\Psi_r(\vec{r}) = C\;\psi_0(\vec{r}) + [\frac{\partial}{\partial\vec{r}}^2 + k^2]^{-1}\;\{(g/|\vec{r}|)\;\Psi_r(\vec{r})\}; \qquad (8)$$

$\psi_0(\vec{r})$ corresponds the solution of the equation:

$$[\frac{\partial}{\partial\vec{r}}^2 + k^2]\,\psi_0(\vec{r}) = 0; \qquad (9)$$

In expression (8), the coefficient C can be any number, including zero. The solution of (9) can be represented in the form of a plane wave: $\psi_0(\vec{r}) = \exp(i\vec{k}\vec{r})$, whose corresponding momentum must satisfy the condition $\vec{k}^2 = k^2$. Since $\psi_0(\vec{r})$ is not related to the interaction, its corresponding plane wave is interpreted as the asymptotic |in> state of the $\Psi_r(\vec{r})$ function and is defined as the limit when $|\vec{r}|\to\infty$ and $V(\vec{r})\to 0$. In some textbooks, the origin of this term is erroneously associated with the general principle of solving an inhomogeneous equation, according to which - if $F_1(x)$ represents a particular solution of the equation:

$$[\frac{\partial}{\partial x}^2 + k^2]\,F(x) = f(x);$$

then the general solution is obtained by the sum $F(x) = F_1(x) + C\,F_0(x)$, where $F_0(x)$ corresponds to the solution of the homogeneous equation obtained for the case $f(x) = 0$. The Schrödinger equation (7) represents a linear homogeneous equation, whose solution is not determined by this rule. In some textbooks, the reason for the origin of this term is indicated as the mathematical algorithm that was introduced in the Lippmann-Schwinger publication (see (Lippmann et al, 1950)). In reality, this argument is not correct either. The authors of this publication attempted to obtain the time-dependent Schrödinger equation by the same variational method, as the "Schwinger-Dyson" equations for Green's functions are obtained in quantum field theory. There is a fundamental difference between these two mathematical algorithms, because the Schwinger-Dyson equations are written with the condition of the existence of a so-called "external source". As a result, inhomogeneous equations are automatically obtained for Green's functions, while the Schrödinger equation represents a homogeneous equation - without the existence of any "external source". Therefore, in the Schwinger-Dyson equations, the Green's function corresponding to the so-called "free particle" appears uniquely defined both in the complete equation and in the iterative series of perturbation theory. While in (8) - an analogous term appears in the form of $C\,\psi_0(\vec{r})$ - that is, not uniquely defined and also allows the possibility of C = 0, which is fundamentally impossible in the case of Schwinger-Dyson equations. Nevertheless, there is still one important detail of similarity between (8) and the Schwinger-Dyson equation - in both equations, if we use the relation (8) multiple times in the term containing the interaction:

$$\Psi_r(\vec{r}) = \psi_0(\vec{r}) + [\frac{\partial}{\partial\vec{r}}^2 + k^2]^{-1}\;\{\psi_0(\vec{r}) + [\frac{\partial}{\partial\vec{r}}^2 + k^2]^{-1}\;\Psi_r(\vec{r})\;\bullet$$
$$(g/|\vec{r}|)\}\;(g/|\vec{r}|) =$$
$$= \psi_0(\vec{r}) + g[\frac{\partial}{\partial\vec{r}}^2 + k^2]^{-1}\{\psi_0(\vec{r})/|\vec{r}|)\} + g^2[\frac{\partial}{\partial\vec{r}}^2 + k^2]^{-1}\bullet$$

$$\{[\frac{\partial}{\partial\vec{r}}^{\,2} + k^2]^{-1}\,[\Psi_r(\vec{r})/|\vec{r}|]/|\vec{r}|\} = \cdots ; \qquad (10)$$

we will get exactly the same iterative series with respect to the whole powers of charge, as we write when solving the Schwinger-Dyson equation in the so-called - "perturbation theory". This series - as a possible method for solving equation (7), without any unnecessary and dubious additional arguments, was first introduced into consideration by M. Born in his 1927 publication (see (Born, 1927)). This method was called the "Born iteration method", in which $\psi_0(\vec{r})$ corresponds to the zero order of iteration, while $\psi_1(\vec{r})$ corresponds to the first order of iteration. $\psi_1(\vec{r})$ is defined as the solution of the equation:

$$[\frac{\partial}{\partial\vec{r}}^{\,2} + k^2]\,\psi_1(\vec{r}) = \psi_0(\vec{r})/|\vec{r}|; \qquad (11)$$

As a rule, for $\psi_0(\vec{r})$, a plane wave parameterization is used:

$$\psi_0(\vec{r}) = \exp(i\vec{p}\vec{r}); \qquad \vec{p}^{\,2} = k^2 ; \qquad (12)$$

However, from general considerations - it would be better to use a parameterization of type (6):

$$\psi_0(\vec{r}) = \int d\vec{p}\,\exp(i\vec{p}\vec{r})\,\delta(k^2 - \vec{p}^{\,2}) ; \qquad (13)$$

But, since the $\hat{H}_0(\vec{r})$ operator commutes with the corresponding momentum operator, no attention is paid to the fact that the full Hamiltonian operator $\hat{H}_r(\vec{r})$ does not commute with this operator and the parameterization indicated in (12) is always used. Using this parameterization, the |in> states of the scattering problem are defined for both particles participating in the collision.

Such an interpretation was introduced in discussions by Born in the above-mentioned work, which was called - "Quantum mechanics of collision processes". In a later published book, titled - "Atomic Physics", the corresponding chapter of the same problem is called "derivation of Rutherford's formula in wave mechanics" (see (Born, 1963)). Presumably, such a transformation from the term "quantum" to "wave" should not only affect the title of the topic and may imply that the author also meant the essence of the topic. The point is that in similar discussions of solving Schrödinger's stationary equation - probability theory is used only at the level of terms - "Ψ-functions, besides their wave nature, also represent probability amplitudes". The generation of the asymptotic |in⟩ state of the scattering problem also corresponds only to an artifact of wave mechanics and has nothing to do with the probabilistic nature of Ψ-functions. Therefore, in existing representations of quantum mechanics, they simply say the phrase: if in the regime of large r, the interaction potential rapidly approaches zero, the solution of equation (7) automatically transitions to the |in⟩ state when $|\vec{r}|\to\infty$. We will address this issue in more detail below.

CHAPTER II: |out> states in the Born iterative scheme

In the mathematical algorithm for the scattering problem, the part that determines the asymptotic |out> state looks even more dubious. This part is realized by those terms of the iterative series that contain interaction potentials. The first such term corresponds to the function $\psi_1(\vec{r})$, which is defined by the relation:

$$\psi_1(\vec{r}) = [\frac{\partial}{\partial\vec{r}}^{\,2} + k^2]^{-1}\,\psi_0(\vec{r})/|\vec{r}| =$$
$$= \int d\vec{r}\,'\,[G(\vec{r} - \vec{r}\,')\exp(i\vec{p}\,\vec{r}\,')]/|\vec{r}\,'| \qquad (14)$$

In the mathematical algorithm for introducing |out⟩ state vectors, the expression of the Green's function G(r-r') in coordinate representation is used, which has the form (see (Schweber, 1961), (Davydov, 1976), (Blokhintsev, 1964)):

$$G_+(\vec{r}) = [\frac{\partial}{\partial\vec{r}}^{\,2} + k^2 + i\epsilon]^{-1}\,\delta^{(3)}(\vec{r}) =$$
$$= -(4\pi)^{-1}\,\frac{\exp[ik|\vec{r}|]}{|\vec{r}|} ; \qquad (15)$$

The superscript "+" indicates the rule for circumventing the pole in the Green's function, which is associated with the interpretation of the "scattered wave". Within this interpretation, |out⟩ states are also defined in the regime of "large $|\vec{r}|$", for which - in the Green's function present in (14), is used a series expansion representation for the expression of the root:

$$|\vec{r} - \vec{r}\,'| = [\vec{r}^{\,2} - 2\vec{r}\vec{r}\,' + \vec{r}\,'^2]^{1/2} = r\,[1 - 2\vec{n}\vec{r}\,'/r + \vec{r}\,'^2/\vec{r}^{\,2}]^{1/2} =$$
$$= r - \vec{n}\vec{r}\,' + O(1/r); \qquad \vec{n} = \vec{r}/|\vec{r}| ; \qquad (16)$$

For the Green's function, authors write the expression:

$$G_+(\vec{r} - \vec{r}\,') = -(4\pi)^{-1}\,\frac{\exp[ik|\vec{r}| - ik\vec{n}\vec{r}\,' + O(1/|\vec{r}|)]}{|\vec{r}|[1+O(1/|\vec{r}|)]} =$$
$$= -(4\pi)^{-1}\,\frac{\exp[ik|\vec{r}| - i\vec{k}\vec{r}\,']}{|\vec{r}|} + O(1/|\vec{r}|^2) ;$$
$$\vec{k} = k\vec{n}; \qquad \vec{k}^{\,2} = k^2; \qquad (17)$$

Since the mentioned Green's function is generated in the expression of $\psi_1(\vec{r})$ containing the interaction, the $\vec{k}$-vector and its corresponding plane wave function are associated with the **|out⟩** state vector - formed by the interaction. The relation $\vec{k}^{\,2} = k^2 = \vec{p}^{\,2}$, is associated with the interpretation of the elastic scattering condition.

After this simplification, the analytical calculation of the corresponding integral is simplified, and using the "elastic scattering" condition, the final answer is written as follows:

$$\psi_{1+}(\vec{r}) = -lim_{\varepsilon\to+0}\,\frac{\exp(ik^+|\vec{r}|)}{|\vec{r}|}\int d\vec{r}\,'\;\frac{\exp(i\vec{p}\vec{r}\,' - i\vec{k}\vec{r}\,')}{|\vec{r}\,'|} =$$
$$= -\frac{\exp(ik|\vec{r}|)}{4\pi k|\vec{r}|sin^2(\frac{\varphi}{2})} + O(1/|\vec{r}|^2);$$
$$cos\varphi = (\vec{p}\,\vec{k})/|\vec{p}|\,|\vec{k}| = (\vec{p}\,\vec{k})/\,k^2 ; \qquad (18)$$

This expression, which is called the Rutherford's formula, is mentioned in many quantum mechanics textbooks. It is necessary to note one technical detail here - the integral indicated in (18) is not uniquely defined in the $|\vec{r}\,'|\to\infty$ limit. Therefore, obtaining the specified result requires the introduction of a certain mathematical recipe. For example, in (Davydov, 1976), this expression is obtained as the limiting case of a screened Coulomb potential V(r) $\sim e^{-|\vec{r}|/r_0}/|\vec{r}|$- when $r_0\to\infty$. When using this potential in (18), the integral

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becomes uniquely defined, and the expression indicated in the answer is obtained quite correctly.

At first glance, it might seem that everything is in order in the mathematical algorithm mentioned above. However, it is not difficult to show that the series expansion method - used to obtain (18), is mathematically incorrect. To demonstrate this, let's rewrite the original expression with a simple transformation of variable as follows:

$$\psi_{1+}(\vec{r}) = \int d\vec{r}' \, [|\vec{r}'||\vec{r}+\vec{r}'|]^{-1} \, e^{ik^+|\vec{r}'|+ \, i\vec{p}(\vec{r}+\vec{r}')} \; ; \qquad (19)$$

where for the parameter k, we have restored the imaginary part, defining by the pole circumvention prescription: $k \to k^+ = (k^2 + i\epsilon)^{1/2} = \{k + i\epsilon/2k + O(\epsilon^2)\}$. This recovery ensures the existence of the integral in the $|\vec{r}'| \to \infty$ limit and also serves as an intermediate regularization, as happens during pole bypassing. This regularization plays the same role in (19) as the $e^{-|\vec{r}|/r_0}$ factor in the screened Coulomb potential in the $r_0 \to \infty$ limit. This complex numerical term $(i\epsilon)$, cannot play the role of regularization in (18) because - unlike (19), where the $(i\epsilon)$ parameter is multiplied by the sign-definite $(i\epsilon|\vec{r}'|)$-factor, due to the use of the series expansion method in (18) - it is multiplied by the sign-indefinite $\vec{n}\vec{r}'$- factor.

If we also use the series expansion method indicated in (17) for $|\vec{r}+\vec{r}'|^{-1}$ in (19), we get:

$$\psi_1(\vec{r}) = -(4\pi)^{-1} \int d\vec{r}' \exp[ik^+|\vec{r}'|+ i\vec{p}(\vec{r}'+\vec{r})] \{1-\vec{n}\vec{r}'/|\vec{r}| + O(1/\vec{r}^2)\} /|\vec{r}||\vec{r}'| =$$

$$= -(4\pi|\vec{r}|)^{-1} [\exp(i\vec{p}\vec{r})] \{1+ (i/|\vec{r}|) \, \vec{n} \, d/d\vec{p} + O(1/\vec{r}^2)\} \cdot \int d\vec{r}' \{\exp[ik^+|\vec{r}'|+ i\vec{p}\vec{r}']\}/|\vec{r}'|; \quad (20)$$

To evaluate this integral, let's switch to spherical coordinates:

$$J = \int d\vec{r}' \exp[ik^+|\vec{r}'|+ i\vec{p}\vec{r}']/|\vec{r}'| =$$

$$= 2\pi \int_0^\infty dr' \, r' \int_{-1}^1 dcos\theta \, \exp[ik^+|\vec{r}'|+ i|\vec{p}||\vec{r}'|cos\theta] =$$

$$= \int_0^\infty dr' \{\exp[i(|\vec{p}| + k^+)r'] - \exp[-i(|\vec{p}| - k^+)r']\} i|\vec{p}|; \quad (21)$$

If we use condition for the elastic scattering - $|\vec{p}|$= k, we will have for the obtained integral:

$$J = \int_0^\infty dr' \, \{ \exp[i2k - \epsilon/2k)r'] - \exp[-\epsilon \, r'/2k]\}/ik =$$

$$= 2i \, [\frac{1}{\epsilon - 4ik^2} - \frac{1}{\epsilon}] \; ; \qquad (22)$$

This expression does not have a limit as $\epsilon \to 0$, which means that the function $\psi_1(\vec{r})$ does not allow expansion in positive integer powers of $(1/|\vec{r}|)$.

A natural question arises - what is the status of the expression indicated in (18), by means of which the differential cross-section of elastic scattering of α-particles is calculated? We will answer this question in the following subsection.

CHAPTER III: The Status of Rutherford's Formula in Wave and Quantum Mechanics

To give a competent answer to the question posed in the previous subsection, it is necessary to try to evaluate the original integral without expanding in powers of $(1/|\vec{r}|)$. This is difficult to implement in coordinate representation because - the Green's function contains a transcendental exponential. Therefore, let's calculate $\psi_1(\vec{r})$ in the momentum representation:

$$\psi_1(\vec{r}) = (2\pi)^{-3} \int d\vec{r}' \int d\vec{q} \, [-\vec{q}^{\,2} + k^2 + i\epsilon]^{-1} \cdot \exp[i\vec{q} \, (\vec{r}- \vec{r}') +i\vec{p} \, \vec{r}']/|\vec{r}'|; \qquad (23)$$

In the resulting expression, the integral over $\vec{r}'$ corresponds to the Fourier transform of the Coulomb potential. The corresponding expression for this transform is easily obtained if we use the integral representation for $1/|\vec{r}'|$:

$$\int d\vec{r}' \exp[i(\vec{p} - \vec{q})\vec{r}'] \, [1/|\vec{r}'|] = \pi^{-1/2} \int_0^\infty dy \, y^{-1/2} \cdot \int d\vec{r}' \exp[i(\vec{p} -\vec{q})\vec{r}' - y\vec{r}'^2 = 4\pi /(\vec{p} - \vec{q})^2; \quad (24)$$

Using the same integral representation, the expression corresponding to the inverse Fourier transform is also easily obtained:

$$\int d\vec{q} \, \exp(i\vec{q}\vec{r})/\vec{q}^2 = \int d\vec{q} \int_0^\infty dy \, \exp(i\vec{q}\vec{r} - y\vec{q}^2) = - 2\pi^2/|\vec{r}|; \quad (25)$$

Accounting for the above, the original integral - without considering numerical coefficients - takes the form:

$$\psi_1(\vec{r}) \sim \int d\vec{q} \, \exp(i\vec{q}\vec{r}) \, [-\vec{q}^{\,2} + k^2 + i\epsilon]^{-1} \, [(\vec{p} - \vec{q})^2]^{-1} =$$

$$= \int d\vec{q} \, \exp[i(\vec{p} - \vec{q})\vec{r}] \, [k^2 - (\vec{p} - \vec{q})^2 + i\epsilon]^{-1} \, [\vec{q}^2]^{-1}; \quad (26)$$

Let's evaluate the behavior of the integrand function in the ultraviolet and infrared regions of momentum values: In the ultraviolet region, i.e. - for large $|\vec{q}|$, we asymptotically have an integral of the type $\int d\vec{q} \, [\vec{q}^2]^{-2}$, which converges. That is - in this region, the original integral is correctly defined. In the infrared region, i.e. - for small $|\vec{q}|$, take into account the condition $\vec{p}^2 = k^2$, we will have $\int d\vec{q} \, [(\vec{q}^2)(\vec{q}\vec{p})]^{-1}$ and the integral is logarithmically divergent. That is, in this region, the original integral is not correctly defined. For a clearer demonstration of this assertion, let's use the same integral representation for the denominators in (26) that we used above:

$$[k^2 - (\vec{p} - \vec{q})^2 + i\epsilon]^{-1} \, [\vec{q}^2]^{-1} = -i[\epsilon + i\vec{q}^2 - 2i\vec{q}\vec{p}]^{-1} \, [\vec{q}^2]^{-1} =$$

$$= -i \int_0^\infty dx \int_0^\infty dy \, \exp[-(\epsilon + i\vec{q}^2 - 2i\vec{q}\vec{p})x - \vec{q}^2 y]; \quad (27)$$

After integrating over the momentum variable, we get:

$$\psi_1(\vec{r}) \sim \int_0^\infty dx \int_0^\infty dy \, \, [\pi/(y + ix)]^{3/2} \cdot \exp\{-[\vec{r}^2 + 4\vec{r}\vec{p} \, x + 4\vec{p}^2 x^2]/4(y+ix)\} \; ; \quad (28)$$

Let's make the transformation: x → xy and integrate over the y-variable - which corresponds to a correctly defined integral:

$$\psi_1(\vec{r}) \sim \pi^2/|\vec{p}| \int_0^\infty dx \, [x(1 + ix)]^{-1} \cdot \exp[-(\vec{r}\vec{p} + \sqrt{\vec{p}^2\vec{r}^2})x/(1+ix)] \; ; \qquad (29)$$

As expected, the resulting integral converges for large x, but is logarithmically divergent for small x.

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After the research carried out above, it is no longer difficult to notice what mathematical detail ensured the derivation of Rutherford's formula in "wave-quantum" mechanics. For this, it will be necessary to return to the expressions indicated in (18) and note the following: In the regime of large $|\vec{r}|$, the series expansion procedure affects two functions - the exponential in the numerator and the root expression in the denominator. The series expansion procedure used for the function in the denominator is in full compliance with the correct mathematical algorithm of the series expansion and therefore - the following expression is completely correct:

$$\{|\vec{r}|[1+0\left(\frac{|\vec{r}'|}{|\vec{r}|}\right)]\}^{-1} = \frac{1}{|\vec{r}|}\{1+ O(|\vec{r}'|/|\vec{r}|^2)\}; \qquad (30)$$

For the exponential function in the numerator, however, the series expansion operation is performed incompletely - only the root in the exponent of the exponential function is expanded in a series, while the expansion of the exponential function itself is not performed. That is, a relation analogous to (30):

$$\exp[ik|\vec{r}| - ik\vec{n}\vec{r}' + 0\left(\frac{1}{|\vec{r}|}\right)] =$$

$$= \exp[ik|\vec{r}| - ik\vec{n}\vec{r}']\{1+ O(|\vec{r}'|/|\vec{r}|^2)\}; \qquad (31)$$

is no longer a correct notation. As a result, the $ik\vec{n}\vec{r}'$- term is retained in the exponent, which ensures obtaining a result different from (22). Therefore, we can conclude:

obtaining Rutherford's formula in "wave" and "quantum" mechanics represents only an artifact corresponding to the incorrect use of mathematical methods.

It is easy to show that a similar result would be obtained if we used the representation indicated in (13) instead of (12) for the $\psi_0(\vec{r})$ function.

Naturally the question arises: In the case of positive eigenvalues of the Hamiltonian operator for relative motion - does the stationary Schrödinger equation have a finite solution?

Let's address this issue in the next subsection.

CHAPTER IV: The Correct Method for Solving the Stationary Equation

To give a competent answer to the question posed in the previous subsection, we need to understand what causes the emergence of infrared divergence in the integral representation of $\psi_1(\vec{r})$. It's easy to notice that this divergence would not occur if we didn't use the relation $\vec{p}^2 = k^2$ in (26). Let's recall where this condition came from: solving the Schrödinger stationary equation in the zeroth order of Born iteration requires the introduction of this condition. Note that - when the Coulomb potential is replaced by a screened Coulomb potential - $V(r) \sim e^{-|\vec{r}|/r_0}/|\vec{r}|$, the same condition no longer causes infrared divergence. That is, the infrared catastrophe is generated when we use the iterative method in the case of the Coulomb potential. Conclusion - In the case of Coulomb interaction, the Born iterative method cannot be used.

Regarding the above conclusion, let's recall the following: We use the same stationary equation to describe bound states with Coulomb potential, but in this case, the energy corresponding to the relative motion of the two-particle system is negative. Solving the corresponding stationary equation limits the allowable numerical set of eigenvalues of the Hamiltonian operator to a discrete spectrum. The numerical values of the spectrum become parametrically dependent on the interaction charges. Also, recall that: to obtain probability amplitudes corresponding to the bound state, the stationary equation must be solved completely. In the case of using the iterative method, the same result cannot be obtained.

It should also be noted here, that:
the explicit forms of these probability amplitude functions do not provide any information about the dynamics of motion of bound particles at discrete energy levels. This is in full agreement with the essence of probabilities - probabilities corresponds to statistical data sets of results of completed events and do not and cannot describe the processes of events. That's why - the solutions of Schrödinger's stationary equation, which - by definition, are objects of probability space, cannot give us information about any chronology of the course of processes and especially - about the dynamics of the course of processes. Based on general considerations of probability theory, neither the probabilities nor probability amplitudes corresponding to the scattering regime should give us information about the chronology of the scattering process itself.

According to current quantum-mechanical concepts - despite the fact that the scattering regime and bound states correspond to the same equation, these equations should still be solved by different mathematical methods, as they correspond to fundamentally different physical phenomena. This view seems logical, since in particle scattering experiments, we prepare the impulses corresponding to the **|in⟩** states of these particles and can assign them any numerical value. Therefore, when solving the corresponding stationary equation, it is automatically assumed that the eigenvalues of the Hamiltonian can neither be discrete nor dependent on the charge of the Coulomb interaction. The bound states and corresponding energies are determined by the process of bound state formation itself, which occurs without our participation. Based on these logical arguments, the abovementioned view seems natural and logical, if not for one question - are the physical phenomena corresponding to these two regimes really fundamentally different, or due to certain preconceptions, are we missing some details when observing them? To clarify this, let's point out the principal differences corresponding to these phenomena:

1. In bound states, discretization of energy levels is observed, which is not observed in the scattering regime;
2. In the scattering regime, interference-diffraction patterns are observed, which are not observed in bound states.

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As recent simple experiments show (see (Peng, 2021), (Part 1. Baghaturia et al, 2025a)), it may turn out that behind these two seemingly entirely different phenomena stands a single common phenomenon, according to which interference patterns represent the same discretization phenomenon as the discretization of energy levels in bound states. The discrete energy levels of a hydrogen-like atom are obtained when we apply the physical requirements of probability normalization. As a result of the mathematical realization of this condition, the numerical values of the energy parameter become dependent on the eigenvalues of the square of the orbital angular momentum corresponding to the relative motion in the bound state. And if these values represent discrete numbers, the energy parameter corresponding to the bound state also becomes discrete.

The orbital angular momentum operator corresponding to relative motion is constructed with the spatial vector and impulses corresponding to this motion and, in its form, fully corresponds to the orbital angular momentum operator of a free particle. Although this operator does not directly correspond to real particles, the finiteness and normalization conditions imposed on the objects of probability space also extend to the eigenfunctions of this operator, because they are used to construct the probabilistic characteristics of real particles. As a result, the eigenvalues of this operator turn out to be just as discrete as in the case of a real quantum object. Note that the equations corresponding to the eigenfunctions and values of both the square of the orbital angular momentum and its third component, have nothing to do with negative numerical values of energy, and the corresponding discrete spectrum is generated in the positive energy regime by itself when the above-mentioned physical conditions are imposed. It is precisely because of this that the energy spectrum of the bound state turns out to be discrete.

On the other hand, quantum objects in the scattering regime also have orbital angular momenta with similar discrete spectra, although no one connects this fact with the so-called "interference" and "diffraction" patterns. As the analysis of the experimental facts indicated in (Peng, 2021) and (Part 1. Baghaturia et al, 2025a) reveals, the formation of "diffraction-interference" patterns obtained on the screen as a result of quantum objects passing through slits occurs not by the mechanism corresponding to the Huygens-Fresnel mathematical principles introduced to describe wave propagation, but by mechanisms of rectilinear motion corresponding to geometric optics. That is, the photons emitted by the laser move in the form of corpuscles, and after colliding with the walls of the holes, the overall image of the traces they leave on the screen, which is mistakenly called the "interference phenomenon", actually has nothing to do with the mechanism of wave propagation determined by the mathematical principle of Huygens-Fresnel.

If we carefully look at the diffraction patterns indicated in (Peng, 2021), (Part 1. Baghaturia et al, 2025a) - which can be easily obtained by anyone - we will also easily notice that these patterns correspond to the phenomenon of spatial discretization, which can only be explained by the scattering of laser beam photons at the boundaries of the slit. Therefore, the question naturally arises - is this spatial phenomenon of discretization not caused by the discretization of the orbital angular momenta of interacting objects participating in the scattering process.

In order to support this idea with mathematical calculations, it is necessary to fully solve the corresponding stationary equation in the same way as is done in the case of bound states. This will not be an easy task, as the slit walls are not simple point-like quantum objects, and the case of photon scattering on them cannot be reduced to the problem of Coulomb scattering - the photon has no charge. However, since a similar pattern should be observed as a result of electrons passing through slits, it will be interesting to fully solve the corresponding equation of Coulomb scattering for charged objects and check - whether we will see any kind of mathematical indication of the aforementioned phenomenon of spatial discretization.

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