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Volatility Estimation in a Jump Di usion Geometric Brownian with Reinvested Dividends and Transaction Cost.

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Abstract

This paper derives a local volatility model for an asset whose dynamics follow a jump di usion geometric Brownian motion, incorporating reinvested divi- dend yields and proportional transaction costs. Building upon the seminal works of Black and Scholes (1973), Merton (1976), and the more recent de- velopments by Opondo et al. (2021, 2025), the study extends Dupire's local volatility framework to accommodate discontinuities and market frictions in asset price behavior. By integrating tools from stochastic calculus, jump pro- cess theory and transaction cost modeling, we derive a modi ed Dupire-type volatility equation tailored for complex nancial environments. This model enhances the accuracy of derivative pricing and risk assessment under more realistic market conditions.

Keywords: *Dupire Volatility, Jump Di usion, Geometric Brownian Motion, Reinvested Dividends, Transaction Costs and Fokker-Planck equation.*

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1. INTRODUCTION

The valuation of nancial derivatives fundamentally depends on the un- derlying dynamics of asset prices. Classical models such as the Black-Scholes framework assume continuous price paths and constant volatility [5]. How- ever, empirical evidence shows that asset prices often exhibit sudden jumps, stochastic volatility, and are in uenced by factors such as dividend payments and transaction costs, factors not adequately captured by standard models.

Merton extended the geometric Brownian motion (GBM) model by incor- porating jump components to address some of these limitations [15]. More recently, researchers such as Opondo, Oduor, and Odundo developed models that integrate reinvested dividends and proportional transaction costs within jump di usion processes, thereby enriching the modeling of asset dynamics in modern nancial markets [16,17].

Local volatility models, introduced independently by Dupire and Derman and Kani in the early 1990s, have become essential tools in option pricing, enabling the calibration of volatility surfaces using market-observed data [10, 11]. However, the classical Dupire equation is limited to pure di usion processes and does not account for jumps, dividends, or market frictions such as transaction costs.

This study aims to derive a modi ed Dupire-type local volatility equa- tion under a Jump Di usion Geometric Brownian Motion (JDGBM) frame- work that incorporates both reinvested dividends and proportional transac- tion costs. The derivation employs tools from stochastic calculus and builds on the theoretical foundation provided by earlier works, including those of Oduor [18] and Li et al. [23], who examined similar features in stochastic environments involving jumps and dividends.

The resulting model enhances the accuracy of volatility estimation, deriva- tive pricing, and hedging strategies in more realistic nancial settings char- acterized by discontinuities and trading frictions.

2. Preliminaries

This study adopts a mathematical modeling framework grounded in stochas- tic calculus to derive a local volatility model for an asset whose price dynamics follow a Jump Di usion Geometric Brownian Motion (JDGBM). The model incorporates two key real-world market features: reinvested dividend yields and proportional transaction costs. The methodology proceeds through the following steps:

2.1. Derivation of the price of a jump di usion Geometric Brownian motion with the re-invested dividend and transaction cost.

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Considering Yin Li et al.[23] and Opondo [17], a jump di usion Geometric Brownian motion with the dividend yielding and transaction cost equation is given by;

$$dS_t = S_t(\mu - \gamma - \tau - \lambda k)dt + S_t\sigma dW_t + S_t dq_t, \quad (1)$$

Where γ is the dividend yielding proportion and τ is the transaction cost pro- portion. By considering the reinvested dividend proportion and transaction cost, the resulting equation from 1 is given by;

$$dS_t = S_t(\mu + \gamma - \tau - \lambda k)dt + S_t\sigma dW_t + S_t dq_t, \quad (2)$$

Equation 2 can be solved using Ito's lemma given by;

$$dY = (\frac{\partial Y}{\partial t} + (\mu+\gamma-\tau-\lambda t)S\frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 Y}{\partial S^2})dt + \sigma S\frac{\partial Y}{\partial S}dW + S\frac{\partial Y}{\partial S}dq_t \quad (3)$$

From equation 3, we want to nd dY where Y is de ned as $ln(S_t)$. First we obtain the partial derivatives involved.

(i). $\frac{\partial Y}{\partial t} = 0$, since Y is not a function of time.

(ii). $\frac{\partial Y}{\partial S} = \frac{1}{S}$, $ln(S)$ with respect to S is $\frac{1}{S}$

(iii). $\frac{\partial^2 Y}{\partial S^2} = \frac{-1}{S^2}$, the second derivative of $ln(S)$

Pluging the values ito's lemma in equation 3

$$dY = (0 + (\mu+\gamma-\tau-\lambda t)S\frac{1}{S} + \frac{1}{2}\sigma^2 S^2(\frac{-1}{S^2}))dt + \sigma S\frac{1}{S}dW + S\frac{1}{S}dq_t \quad (4)$$

Simplifying equation 4, we obtain;

$$dY = ((\mu+\gamma-\tau-\lambda t) - \frac{1}{2}\sigma^2)dt + \sigma dW + dq_t \quad (5)$$

Since $Y = ln(S_t)$, we have;

$$d(ln(S_t)) = (\mu+\gamma-\tau-\lambda t - \frac{1}{2}\sigma^2)dt + \sigma dW + dq_t \quad (6)$$

Integrating equation 6 with respect to t

$$ln(S_t) - ln(S_0) = (\mu+\gamma-\tau-\lambda t - \frac{1}{2}\sigma^2)t - t_0 + \sigma W_t + q_t \quad (7)$$

Simplifying further equation 7, we obtain;

$$ln(\frac{S_t}{S_0}) = (\mu+\gamma-\tau-\lambda t - \frac{1}{2}\sigma^2)t - t_0 + \sigma W_t + q_t \quad (8)$$

Taking exponential of both sides from equation 8, we obtain;

$$S_t = S_0 \exp^{(\mu+\gamma-\tau-\lambda t-\frac{1}{2}\sigma^2)t - t_0 + \sigma W_t + q_t} \quad (9)$$

3. Derivation of Fokker-Planck equation

The Fokker-Planck equation describes the time evolution of the probabil- ity density function of the position of a particle under the in uence of forces and random perturbations.

Step 1: Start with the Stochastic Di erential Equation Consider the SDE for a stochastic process X(t):

$$dX(t) = \mu(X,t)\,dt + \sigma(X,t)\,dW(t)$$

where:

$\mu(X,t)$ is the drift term,

$\sigma(X,t)$ is the di usion term,

$W(t)$ is a Wiener process (Brownian motion).

Step 2: Write the Corresponding Fokker-Planck Equation The Fokker-Planck equation describes the time evolution of the probability density function P (x, t) of the stochastic variable X(t).

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x,t)P(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2(x,t)P(x,t) \quad (11)$$

This equation can be derived using the It calculus and the concept of the probability density function.

Step 3: Derivation Using the Probability Density Function Probability Density Function Evolution

Let P (x, t) be the probability density function of the stochastic variable X(t). The probability that X(t) lies between x and x + dx at time t is P (x, t) dx. Master Equation Approach Consider the change in P (x, t) over an in nitesimal time interval dt:

$$P(x, t+dt) = \int_{-\infty}^{\infty} P(x', t)w(x' \to x, dt)\,dx' \quad (12)$$

where $w(x' \to x, dt)$ is the transition probability from x' to x in time dt.

Transition Probability Expansion

Assume the transition probability can be expanded in a Taylor series around x':

$$w(x' \to x, dt) \approx \delta(x' - x) + \sum_{n=1} \frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n}\delta(x' - x)\ \langle(X(t+dt) - X(t))^n\rangle \quad (13)$$

First and Second Moments

The rst and second moments of the increments are given by:

$$\langle X(t+dt) - X(t)\rangle = \mu(x,t)\,dt \quad (14)$$

$$\langle(X(t+dt) - X(t))^2\rangle = \sigma^2(x,t)\,dt \quad (15)$$

Integrate and Simplify

Substitute these moments into the master equation and integrate by parts:

$$P(x, t+dt) - P(x,t) = \int_{-\infty}^{\infty} P(x',t)\ \delta(x'-x)\mu(x',t)\,dt + \frac{1}{2}\delta''(x'-x)\sigma^2(x',t)\,dt\ dx' \quad (16)$$

Expanding and simplifying gives:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x,t)P(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2(x,t)P(x,t) \quad (17)$$

Step 4: Final Fokker-Planck Equation

Combining the terms, the Fokker-Planck equation is:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x,t)P(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2(x,t)P(x,t) \quad (18)$$

This equation describes the evolution of the probability density function $P(x,t)$ for a stochastic process $X(t)$ governed by the SDE with drift $\mu(x,t)$ and di usion $\sigma(x,t)$.

(10)

4. Dupire's Volatility equation

The local volatility model was introduced by Dupire[11] and Derman[10]. This has become one of the most extensively used models in pricing of deriva- tives across asset classes. The Dupire equation enables us to determine the volatility function in a local volatility model from quoted call and put options in the market. The price dynamics in the local volatility model under the risk neutral measure are given by;

$$dS(t) = (r(t) - q(t)S(t))dt + \sigma(S,t)S(t)dZ(t), \quad (19)$$

Where r(t) is the risk-free interest rate, q(t) is a continuous dividend yield at time t, σ(S, t) is the volatility and Z(t) is the

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Wienner process. The Black-Scholes partial di erential for any contingent claim f (S, t) is given as;

$$\frac{\partial f(S,t)}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 f(S,t)}{\partial S^2} + (r(t) - q(t))S\frac{\partial f(S,t)}{\partial S} - r(t)f(S,t) = 0, \quad (20)$$

Where r(t) is the risk-free interest rate in the market. The t dependence of r(t) is the `term structure' of interest rates.

Suppose that a European call option at time t, priced at a discounted ex- pectation where C(T, t) denotes the discounting factor of a risk-zero coupon bond, then it can be shown that;

$$h(S,t) = C(T,t)\int_K^{\infty} max((S(T) - K)\phi(S(T), S(t)))dS(T), \quad (21)$$

where $\phi(S(T), S(t))$ is the probability density function of conditional on S(t). If we di erentiate equation 21 twice with respect to K, we obtain;

$$\phi(S(T), S(t)) = K = \frac{\partial^2 h(S,t)}{\partial K^2}C(T,t)^{-1} \quad (22)$$

These relations 22 are known as the Breedon-Litzenberger formulas.

By using Kolmogorov's forward equation for transitional probability density function $\phi(S(T), S(t))$ the Fokker-Plank equation on

$$dS(t) = \mu(S,t)S(t)dt + \sigma(S,t)S(t)dZ(t), \quad (23)$$

we obtain the density of the random variable S(T)

$$\frac{\partial\phi(S(T), S(t))}{\partial T} - \frac{1}{2}\frac{\partial^2 h(S,t)}{\partial S(T)^2}(\sigma(S(T),T)^2 S(T),T)^2\phi(S(T),S(t))) + \frac{\partial h(S,t)}{\partial S(T)}\mu(S(T),T)\phi(S(T),S(t)) = 0, \quad (24)$$

where the function $\phi(S(T), S(t))$ is the density of random variable S(T) and time T conditional on the initial value S(t) and the model 23. If this density function is known and di erentiable with respect to time at maturity T , the drift rate and the volatility terms in the related stochastic process are those that solve equation 24. On the other hand if the drift rate and volatility terms of the the stochastic process are known, the solution to the partial di erential equation 24 is the density function. It can be observed that the density function in equation 22 is expressed in terms of the strike price K. Therefore if di erentiation is taken with respect to K, with drift and volatility functions evaluated at K, equation 24 can be re-written regarding h(S, t) as a function of strike price as;

$$\frac{\partial K}{\partial T} - \frac{1}{2}\frac{\partial^2 h}{\partial K^2}[\sigma(K,T)^2K^2\phi(K)] + \frac{\partial h}{\partial K}[(r(T) - q(T))K\phi(K)] = 0 \quad (25)$$

Using equation 22 and substituting $\phi(K)$ in the rst term of equation 25 we obtain;

$$\frac{\partial}{\partial T}[C(T,t)^{-1}(\frac{\partial^2 h}{\partial K^2})] - \frac{1}{2}\frac{\partial^2 h}{\partial K^2}[\sigma(K,T)^2K^2\phi(K)] + \frac{\partial h}{\partial K}[(r(T) - q(T))K\phi(K)] = 0 \quad (26)$$

Di erentiating the rst term of equation 26 with respect to T using the chain rule and then expanding we get;

$$C(T,t)^{-1}\frac{\partial}{\partial T}(\frac{\partial^2 h}{\partial K^2}) + r(T)C(T,t)^{-1}\frac{\partial^2 h}{\partial K^2} + \frac{\partial h}{\partial K}[(r(T) - q(T))K\phi(K)] - \frac{1}{2}\frac{\partial^2 h}{\partial K^2}[\sigma(K,T)^2K^2\phi(K)] = 0 \quad (27)$$

Integrating equation 27 once, multiplying by C(T, t) and substituting for

$\phi(K)$ we have;

$$\frac{\partial h}{\partial K}(\frac{\partial h}{\partial T}) + r(T)\frac{\partial h}{\partial K} + (r(T) - q(T))K\frac{\partial^2 h}{\partial K^2} - \frac{1}{2}\frac{\partial h}{\partial K}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}] = a(T), \quad (28)$$

where a(T) is a constant of integration. Integrating 28 with respect to K we obtain;

$$\frac{\partial h}{\partial T} + r(T)h + [r(T) - q(T)]K\frac{\partial h}{\partial K} - r(T)h + q(T)h - \frac{1}{2}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}] = a(T)K + b(T), \quad (29)$$

where b(T) is a constant of integration that relates to second integration. When we re-arrange equation 29 and simplify we have;

$$\frac{\partial h}{\partial T} + [r(T) - q(T)]K\frac{\partial h}{\partial K} + q(T)h - \frac{1}{2}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}] = a(T)K + b(T) \quad (30)$$

According to Dupire[11], it is assumed that all the terms on the left hand side of equation 30 decay when K tends to $+\infty$ so that $a(T) = b(T) = 0$. This implies that equation 30 will become;

$$\frac{\partial h}{\partial T} + [r(T) - q(T)]K\frac{\partial h}{\partial K} + q(T)h - \frac{1}{2}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}] = 0, K > 0 \quad (31)$$

This gives the price of a European option expressed as a function of T and K when t and S are xed. To get the volatility function we re-arrange equation 31 as

$$\sigma(K,T)^2 = 2\left(\frac{\frac{\partial h}{\partial T} + (r(T) - q(T))K\frac{\partial h}{\partial K} + q(T)h}{K^2\frac{\partial^2 h}{\partial K^2}}\right) \quad (32)$$

or

$$\sigma = \sqrt{2\left(\frac{\frac{\partial h}{\partial T} + (r(T) - q(T))K\frac{\partial h}{\partial K} + q(T)h}{K^2\frac{\partial^2 h}{\partial K^2}}\right)} \quad (33)$$

Equation 33 de nes the value of volatility of an option at time T and the strike price K when dividend is paid out. It is called Dupire's volatility equation.

5. Results

This section presents the results obtained from applying the jump di u- sion geometric Brownian motion model with reinvested dividends and trans- action costs for volatility estimation. The model's extensions introduce mar- ket frictions and discontinuities that signi cantly in uence the behavior of asset price dynamics. The estimated volatility re ects more realistic market conditions, showing that traditional models may underestimate risk in the presence of these additional factors. These results provide a re ned perspec- tive for pricing derivatives and managing risk in markets exposed to abrupt shifts and trading frictions.

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5.1. Derivation of Dupire volatility equation with the re-invested dividend yielding asset and Transaction cost.

Using the spirit of Dupire, we derive a volatility equation of the asset price that follows a jump di usion Geometric Brownian motion with re-invested dividends and transaction cost.

$$dS_t = (\mu + \gamma - \tau - \lambda k)S_t dt + \sigma S_t dW_t + S_t dq_t, \qquad (34)$$

where μ is the growth rate, σ is the volatility, λ is the rate at which the jumps happen, k is the average jump size measured as a proportional increase in asset price qt is the poisson process with intensity of λ [17]. The aim is to show that there is a unique volatility function σ(S, t).

Dupire equation needs to be extended to incorporate re-inested dividend, transaction cost and the jump component. If we apply the black-Scholes Merton PDE for any claim of asset value f (S, t), we have

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r(t) + \gamma - \tau - \lambda k)S\frac{\partial f}{\partial S} + r\frac{\partial f}{\partial S}S - rf = 0, \qquad (35)$$

where r(t) is the risk-free interest rate in the market since we are dealing with price derivatives. If we consider a European call option the process of nding a fair option value of f (S, t), will depend on asset price S(t) and time t. Therefore the function f (S, t) can be written for the value of the contract with boundary condition

$$f(S,t) = max(S(T) - K, 0) \qquad (36)$$

At a time t before expiry date the price of the put option will be a function of $S(t)$, t, T. De ning $\varphi(\frac{S(T)}{S(t)})$ as the pdf condition on S(t) then 36 can be written as;

$$f(S,t) = P(t,T)\int_K^\infty max(S(t) - K, 0)\varphi(\frac{S(T)}{S(t)})dS(T) \qquad (37)$$

Di erentiating 37 twice with respect to K we get;

$$\varphi(\frac{S(T)}{S(t)}) = K = \frac{\partial^2 f(S,t)}{\partial K^2}P(t,T)^{-1} \qquad (38)$$

Applying the Fokker-Plank equation and using Kolmogorov's foward equa- tion on 34 we obtain

$$\frac{\partial\varphi(\frac{S(T)}{S(t)})}{\partial T} - \frac{1}{2}\frac{\partial^2 f(S,t)}{\partial S(T)^2}[\sigma^2 S^2\varphi(\frac{S(T)}{S(t)}) + S(t)\varphi(\frac{S(T)}{S(t)})dq_t] + \frac{\partial f(S,t)}{\partial S(T)}(\mu + \gamma - \tau - \lambda k)S(t)\varphi(\frac{S(T)}{S(t)}) = 0 \qquad (39)$$

Using the approach of Dupire[11] taking f as a function of strike price K in equation 39, with di erentiation taken with respect to drift and volatil- ity function evaluated at K (because the density function in equation 38 is expressed in terms of K. Equation 39 can be re-written as;

$$\frac{\partial K}{\partial T} - \frac{1}{2}\frac{\partial^2 f(S,t)}{\partial K^2}[\sigma^2 K^2\varphi(K) + K\varphi(K)dq_t] + \frac{\partial f(S,t)}{\partial k}(\mu + \gamma - \tau - \lambda k)K\varphi(K) = 0 \qquad (40)$$

Using equation 38 and substituting for φ(K) in the rst term of equation 40 we have;

$$\frac{\partial}{\partial T}[P(t,T)^{-1}\frac{\partial^2 f}{\partial K^2}] - \frac{1}{2}\frac{\partial^2 f(S,t)}{\partial K^2}[\sigma^2 K^2\varphi(K) + K\varphi(K)dq_t] + \frac{\partial f(S,t)}{\partial K}(\mu + \gamma - \tau - \lambda k)K\varphi(K) = 0 \qquad (41)$$

Using chain rule to di erentiate the rst term of 41 with respect to T and expand we get

$$[P(t,T)]^{-1}\frac{\partial}{\partial T}(\frac{\partial^2 f}{\partial S^2}) + \mu(t)[P(t,T)]^{-1}\frac{\partial^2 f}{\partial K^2} - \frac{1}{2}\frac{\partial^2 f(S,t)}{\partial K^2}[\sigma^2 K^2\varphi(K) + K\varphi(K)dq_t] + \frac{\partial f(S,t)}{\partial K}(\mu + \gamma - \tau - \lambda k)K\varphi(k) = 0 \qquad (42)$$

substituting for φ(K), multiplying by P (t, T) then integrating once w.r.t K
we have

$$\frac{\partial}{\partial T}(\frac{\partial f}{\partial K}) + \mu(t)\frac{\partial f}{\partial S} - \frac{1}{2}\frac{\partial f(S,t)}{\partial k}[\sigma^2 K^2\varphi(K) + K\varphi(K)dq_t]\frac{\partial^2 f}{\partial K^2} + (\mu + \gamma - \tau - \lambda k)K\frac{\partial^2 f}{\partial K^2} = \alpha(T), \qquad (43)$$

where α(T) is the constant of integration. integrating again w.r.t K we get

$$\frac{\partial f}{\partial T} + \mu(t)f + (\mu + \gamma - \tau - \lambda k)K\frac{\partial f}{\partial K} - \mu(t)f - \gamma(t)f + \tau(t)f + \lambda k(t)f - \frac{1}{2}[\sigma^2 K^2\varphi(K) + K\varphi(K)q_t]\frac{\partial^2 f}{\partial K^2} = \alpha(T)S + \beta(T), \qquad (44)$$

where $\beta(T)$ is a constant of integration relating to the second integration. As per the approach of Dupire[11], it is assumed that all the terms on the left hand side of equation 44 decay when K tends to∞ so that $\alpha(T) = \beta(T) = 0$ hence 44 becomes

$$\frac{\partial f}{\partial T} + (\mu + \gamma - \tau - \lambda k)K\frac{\partial f}{\partial K} - \gamma(t)f + \tau(t)f + \lambda k(t)f - \frac{1}{2}[\sigma^2 K^2\varphi(K) + K\varphi(K)q_t]\frac{\partial^2 f}{\partial K^2} = 0 \qquad (45)$$

; K > 0 This is the price of a European option expressed as a function of T and K (For xed T and K). Rearranging equation 45, we get;

$$\sigma^2 = 2\left(\frac{\frac{\partial f}{\partial T} + (\mu + \gamma - \tau - \lambda k)K\frac{\partial f}{\partial K} - \gamma(t)f + \tau(t)f + \lambda k(t)f + \frac{1}{2}K\varphi(K)q_t}{K^2\varphi(K)\frac{\partial^2 f}{\partial K^2}}\right) \qquad (46)$$

Finding the square root on both sides of equation 46 we obtain;

$$\sigma = \sqrt{2\left(\frac{\frac{\partial f}{\partial T} + (\mu + \gamma - \tau - \lambda k)K\frac{\partial f}{\partial K} - \gamma(t)f + \tau(t)f + \lambda k(t)f + \frac{1}{2}K\varphi(K)q_t}{K^2\varphi(K)\frac{\partial^2 f}{\partial K^2}}\right)} \qquad (47)$$

Equation 47 is the volatility model with;

1. $\frac{\partial f}{\partial T} = \vartheta = -KN'(d_1)\frac{\sqrt{\sigma}}{2\sqrt{T-t}} - rS^* e^{-r(T-t)}N(d_2)$
   is known as `Theta'
2. $\frac{\partial f}{\partial K} = \Delta = N(d_1)$
   is known as `Delta'

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3. $\frac{\partial^2 f}{\partial K^2} = \Gamma = \frac{N'(d_1)}{K\sigma\sqrt{T-t}}$ is known as 'Gamma'

4. $N(x) = \frac{1}{\sqrt{2\Pi}} \int e^{-\frac{x^2}{2}} dx$

5. $N'(d_1) = \frac{\partial N(d_1)}{\partial d_1} = \frac{1}{\sqrt{2\Pi}} e^{-\frac{d_1^2}{2}}$

Given that

i. $d_1 = \frac{\log(\frac{S}{K}) + (\mu + \nu - \tau - \lambda k + \frac{\sigma^2}{2})T - t}{\sigma \sqrt{T - t}}$

ii. $d_2 = d_1 - \sigma\sqrt{T - t}$

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