Global Journal of Engineering and Technology [GJET].
ISSN: **2583-3359** (Online)
Frequency: Monthly
Published By GSAR Publishers
Journal Homepage Link- https://gsarpublishers.com/journal-gjet-home/

OPEN ACCESS

Some Transformations of q - Hypergeometric Series

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Article History

Received: 15/02/2025
Accepted: 22/02/2025
Published: 24/02/2025

Vol – 4 Issue – 2

PP: - 01-02

Abstract
In this paper, we have used a series identity, given by Fine to produce some new results of basic hypergeometric series. Further, we have investigated some special cases of our results.

Key Words: *Basic (or $q-$) Hypergeometric series, Basic Hypergeometric sum- mations and transformations, Identity.*

AMS Subject Classification 2010: *33D15.*

1 Introduction

We define a basic (or $q-$) hypergeometric series as [2]

$$ {}_r\phi_s\left(a_1,a_2,...,a_r;b_1,b_2,...,b_s;q,t\right) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n}{(q,b_1,b_2,...,b_s;q)_n}[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r}t^n, \quad (1.1)$$

where

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

We can see [2] for the convergence conditions of basic hypergeometric series. Fine [3] established a remarkable and useful identity which is given below.

If

$$g(t) = \sum_{n=0}^{\infty} A_n t^n, \quad (1.2)$$

then

$$\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(bq;q)_n}A_n t^n = \frac{(aq;q)_\infty}{(bq;q)_\infty)}\sum_{k=0}^{\infty}\frac{(b/a;q)_k}{(q;q)_k}(aq)^k g(tq^k). \quad (1.3)$$

The following known results are also required in our work, in the next section.

$$ {}_3\phi_2(a,b,e;aq,de;q,t)\Big) = \frac{(bt/q,e,deq/b;q)_\infty}{(t,eq/b,de;q)_\infty}\times {}_3\phi_2(q,aq/b,q/b;aq,deq/b;q,bt/q). \quad (1.4)$$

(Equ 2.2 [5])

$$ {}_6\phi_5(a,q\sqrt{a},-q\sqrt{a},b,c,a/bc;\sqrt{a},-\sqrt{a},aq/b,aq/c,bcq;q,tq) = (1-t){}_4\phi_3(aq,bq,cq,aq/bc;aq/b,aq/c,bcq;q,t). \quad (1.5)$$

(Equ. 4.22 [1])

$$ {}_2\phi_1(a,b;e;q,t) = \frac{(b,at;q)_\infty}{(t,e;q)_\infty}{}_2\phi_1(e/b,t;at;q,b). \quad (1.6)$$

(Equ. 2.2a [6])

$$\sum_{n=0}^{\infty}\frac{(bq,cq,q/bc;q)_n}{(b,c,bcq;q)_n}t^n = 1+\frac{t}{bc}\frac{(b-q)(c-q)}{(1-b)(1-c)}\sum_{n=0}^{\infty}\frac{(q/bc,q^2/b,q^2/c;q)_n}{(q/b,q/c,bcq;q)_n}t^n. \quad (1.7)$$

(Equ. 2.4.14 [4])

2. Main Results

In this section, we have developed some unilateral results of basic hypergeomet- ric series.

$$\sum_{k=0}^{\infty}\frac{(b/a,t;q)_k}{(bt,q;q)_k}(aq)^k{}_3\phi_2(q,aq/b,q/b;aq,deq/b;q,btq^{k-1}) = \frac{(bq,t,eq/b,de;q)_\infty}{(aq,bt,e,deq/b;q)_\infty}{}_3\phi_2(a,b,e;bq,de;q,t). \quad (2.1)$$

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$$\sum_{k=0}^{\infty}\frac{(b/a;q)_k}{(q;q)_k}(aq)^k{}_4\phi_3(aq,bq,cq,aq/bc;aq/b,aq/c,bcq;q,tq^k)$$
$$-t\sum_{k=0}^{\infty}\frac{(b/a;q)_k}{(q;q)_k}(aq^2)^k{}_4\phi_3(aq,bq,cq,aq/bc;aq/b,aq/c,bcq;q,tq^k)$$
$$=\frac{(bq;q)_\infty}{(aq;q)_\infty}{}_7\phi_6(a,q\sqrt{a},-q\sqrt{a},b,c,a/bc,aq;\sqrt{a},-\sqrt{a},aq/b,aq/c,bcq,bq;q,tq). \tag{2.2}$$

$$\sum_{k=0}^{\infty}\frac{(b/a,t;q)_k}{(q,at;q)_k}(aq)^k{}_2\phi_1(e/b,tq^k;atq^k;q,b)$$
$$=\frac{(bq,t,e;q)_\infty}{(aq,at,b;q)_\infty}{}_3\phi_2(a,b,aq;e,bq;q,t). \tag{2.3}$$

$$\sum_{k=0}^{\infty}\frac{(b/a;q)_k}{(q;q)_k}(aq)^k+t\frac{(1-q/b)(1-q/c)}{(1-b)(1-c)}\times$$
$$\sum_{k=0}^{\infty}\frac{(b/a;q)_k}{(q;q)_k}(aq^2)^k{}_4\phi_3(q,q/bc,q^2/b,q^2/c;q/b,q/c,bcq;q,tq^k)$$
$$=\frac{(bq;q)_\infty}{(aq;q)_\infty}{}_4\phi_3(q,aq,cq,q/bc;b,c,bcq;q,t). \tag{2.4}$$

Proof of (2.1): Taking

$$A_n=\frac{(a,b,e;q)_n}{(aq,de,q;q)_n} \tag{2.5}$$

and putting in (1.2) and applying (1.4), we obtain

$$g(t)=\sum_{n=0}^{\infty}\frac{(bt/q,e,deq/b;q)_\infty}{(t,eq/b,de;q)_\infty}{}_3\phi_2(q,aq/b,q/b;aq,deq/b;q,bt/q). \tag{2.6}$$

Next, using (2.5) and (2.6) in (1.3), we get (2.1). Proof of (2.2): If

$$A_n=\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a/bc;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bcq;q)_n}q^n \tag{2.7}$$

substituting in (1.2) and using (1.5), we have

$$g(t)={}_4\phi_3(aq,bq,cq,aq/bc;aq/b,aq/c,bcq;q,t)$$
$$-t{}_4\phi_3(aq,bq,cq,aq/bc;aq/b,aq/c,bcq;q,t). \tag{2.8}$$

After substituting (2.7) and (2.8) in (1.3) and we obtain (2.2). Proof of (2.3): On taking

$$A_n=\frac{(a,b;q)_n}{(e,q;q)_n}, \tag{2.9}$$

putting in (1.2) and applying (1.6), we get

$$g(t)=\frac{(b,at;q)_\infty}{(t,e;q)_\infty}{}_2\phi_1(e/b,t;at;q,b). \tag{2.10}$$

Now, putting (2.9) and (2.10) in (1.3), we obtain (2.3). Proof of (2.4): Choosing

$$A_n=\frac{(bq,cq,q/bc;q)_n}{(b,c,bcq;q)_n} \tag{2.11}$$

and substituting it in (1.2) and using (1.7), we have

$$g(t)=1+\frac{(1-q/b)(1-q/c)}{(1-b)(1-c)}\sum_{n=0}^{\infty}\frac{(q/bc,q^2/b,q^2/c;q)_n}{(q/b,q/c,bcq;q)_n}t^{n+1}. \tag{2.12}$$

After substituting (2.11) and (2.12) in (1.3), we obtain (2.4).

3. Some Special Cases

Putting $b=q$ and $e=0$ in (2.1), we get

$${}_2\phi_1(q/a,t;qt;q,aq)=\frac{(q^2,t;q)_\infty}{(aq,qt;q)_\infty}{}_2\phi_1(a,q;q^2;q,t). \tag{3.1}$$

If a = 1 in (3.1) and it becomes

$${}_2\phi_1(q,t;qt;q,q)=\frac{(q^2,t;q)_\infty}{(q,qt;q)_\infty}. \tag{3.2}$$

On substituting c → ∞ in (2.2), we have

$$\sum_{k=0}^{\infty}\frac{(b/a;q)_k}{(q;q)_k}(aq)^k{}_2\phi_1(aq,bq;aq/b;q,tq^k/b)$$
$$-t\sum_{k=0}^{\infty}\frac{(b/a;q)_k}{(q;q)_k}(aq^2)^k{}_2\phi_1(aq,bq;aq/b;q,tq^k/b)$$
$$=\frac{(bq;q)_\infty}{(aq;q)_\infty}{}_5\phi_4(a,q\sqrt{a},-q\sqrt{a},b,aq;\sqrt{a},-\sqrt{a},aq/b,bq;q,t/b). \tag{3.3}$$

On substituting e = b in (2.3), we obtain

$${}_2\phi_1(b/a,t;at;q,aq)=\frac{(bq,t;q)_\infty}{(aq,at;q)_\infty}{}_2\phi_1(a,aq;bq;q,t). \tag{3.4}$$

Taking t = b/a2 in above, and it gives a q-binomial theorem

$${}_1\phi_0(b/a^2;q;aq)=\frac{(bq/a;q)_\infty}{(aq;q)_\infty}. \tag{3.5}$$

If c = q2 in (2.4), we get

$${}_4\phi_3(q,aq,q^3,1/bq;b,q^2,bq^3;q,t)=1+t\frac{(1-q/b)(1-1/q)}{(1-b)(1-q^2)}\frac{(bq^2,aq;q)_\infty}{(aq^2,bq;q)_\infty}. \tag{3.6}$$

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