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The Brezis-Nirenberg problem with asymptotic analysis of its low energy sign-changing solutions in theory of Sobolev spaces

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Abstract
One of the problems in theory of Sobolev spaces is the Brezis-Nirenberg problem which is appeared in 1983 by Brezis and Nirenberg. In this paper we study the Brezis-Nirenberg problem which is given by the form

$$(P_\varepsilon) \quad \begin{cases} -\Delta u = |u|^{4/(n-2)}u + \varepsilon u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ε is a real positive parameter and Ω is a smooth bounded domain in R^n, $n \geq 3$. That is, in this paper, for a family of sign-changing solutions with two bubbles, we introduce and investigate some asymptotic analysis.

Key words: *Sobolev space; smooth bounded domain; asymptotic analysis; sign-changing solution.*
AMS classification: *Primary 46E35, 37C60, 35B40*

1. INTRODUCTION

The Brezis-Nirenberg problem is introduced first fundamental results about the existence of positive solutions were obtained by H. Brezis and L. Nirenberg in 1983. The authors explain in [13] that dimension plays a crucial role in the study of (P_ε). They proved that if $n \geq 4$ there exists a positive solution of (P_ε) for every $\varepsilon \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ being the first eigenvalue of $-\triangle$ in Ω with Dirichlet boundary conditions. While for $n = 3$, there are positive solutions only for $\varepsilon \in (\lambda^*, \lambda_1)$, where $\lambda^* := \lambda^*(\Omega)$ is a positive constant dependent on Ω. Iacopetti et al., [21], considered the classical Brezis-Nirenberg problem in the unit ball of $\mathbb{R}^N$, $N \geq 3$ and analyze the asymptotic behavior of nodal radial solutions in the low dimensions $N = 3, 4, 5, 6$ as the parameter converges to some limit value which naturally arises from the study of the associated ordinary differential equation. Cora et al., [22], studied the asymptotic and qualitative properties of least energy radial sign-changing solutions of the fractional BrezisNirenberg problem ruled by the s-Laplacian, in a ball of $\mathbb{R}^n$ and showed that they change sign at most twice and their zeros coincide with the sign-changes. Iacopetti et al., [23] , showed that there exists a sign-changing solution whose positive part concentrates and blows-up at the center of symmetry of the domain, while the negative part vanishes. This research project is based on [12]. Precisely, we study the following semi-linear elliptic problem:

*Corresponding Author: Ayidh Munahi Bati Alqahtani

$$(P_\varepsilon) \quad \begin{cases} -\Delta u = |u|^{p-1}u + \varepsilon u & \text{in } \Omega, \quad \text{(a)} \\ u = 0 & \text{on } \partial\Omega, \quad \text{(b)} \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n, n \geq 3, p+1 = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$ and ε is a real positive parameter. Concerning the case of sign-changing solutions, the existence results hold for $n \geq 4$ both for $\varepsilon \in (0, \lambda_1(\Omega))$ and $\varepsilon > \lambda_1(\Omega)$ as shown in [15, 16]. Furthermore, in [18], the authors proved that, if Ω is symmetric and $n = 4, 5$, there exists a sign-changing solution whose positive part concentrates and blows-up at the center of symmetry of the domain, while the negative part vanishes, as $\varepsilon \to \lambda_1(\Omega)$. Note that the small dimensions $n = 4, 5, 6$ are specific to this problem. Indeed, Atkinson, Brezis and Peletier show in [19] that if Ω is a ball, then there exists $\tilde{\lambda} := \tilde{\lambda}(n)$ so there are no radial sign-changing solutions of (P_ε) for $\varepsilon \in (0, \tilde{\lambda})$. While, in [20], the authors gave asymptotic profile of the positive and negative part of radial solution u_ε in dimensions $n = 3, 4, 5, 6$ as ε tends to some limit value. However, for $n \geq 7$, Schechter and Zou have shown in [6] that in any bounded smooth domain, there is an infinity of sign-changing solutions for any $\varepsilon > 0$. Concerning the low energy sign-changing solutions of (P_ε), a study has been carried out in [7] concerning the solutions u_ε satisfying $\frac{1}{c_1} \leq -\frac{\max u_\varepsilon}{\min u_\varepsilon} \leq c_1$. The authors were able to prove the axial symmetry results for the same kinds of solutions in a ball. Next, A. Iacopetti and G. Vaira built in [8] solutions in the form of: $u_\varepsilon = \delta_{a,\lambda_1} - \delta_{a,\lambda_2} + v_\varepsilon$ with $\lambda_1/\lambda_2 \to 0$ or $+\infty$, where

$$\delta(x) := \delta_{a,\lambda}(x) = c_0 \frac{\lambda^{(n-2)/2}}{(1 + \lambda^2|x-a|^2)^{(n-2)/2}}, \quad \lambda > 0, \quad a \in \mathbb{R}^n, \tag{2}$$

$c_0 := (n(n-2))^{\frac{n-2}{4}}$, describe all regular positive solutions of the Yamabe problem $-\Delta u = u^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$. This result has been proved only for large dimensions $n \geq 7$. Note that the size $n \geq 7$ is optimal, since in [9], Iacopetti and Pacella showed that, in dimension $n = 4, 5, 6$, the sign-changing solutions of the form above do not exist in any bounded smooth domain. In [9], the authors have imposed $a_1 = a_2$, this choice of points is compulsory for their argument based on the Pohazaev identity. In this project, we have considered a general case of low energy sign-changing solutions whose the positive and the negative parts blow-up with different speeds. This kind of solutions u_ε have to satisfy

$$\|u_\varepsilon\|^2 := \int_\Omega |\nabla u_\varepsilon|^2 \to 2S_1^{n/2}, \quad \text{and} \quad \max u_\varepsilon / \min u_\varepsilon \to 0 \quad \text{or} \quad -\infty, \quad \text{as } \varepsilon \to 0, \tag{3}$$

where S_1 is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, that is,

$$S_1 := \inf \left\{ \frac{\|u\|^2_{H_0^1(\Omega)}}{\|u\|^2_{L^{2n/(n-2)}(\Omega)}}, u \in H_0^1(\Omega), u \neq 0 \right\}.$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Li et al., [2], extended and improved the existence of sign-changing solutions established by some researchers. Ben Ayed et al., [3], studied low energy sign changing solutions of some critical exponent problem on a smooth bounded domains and prove some comparison results among some limit values of some parameters which are related to the existence of positive or of sign changing solutions. Liu and Xiaolong [4] developed the limit profiles for the symmetric Palais-Smale sequence by the concentration compactness principle and concluded that the problem admits an odd solution with some nodal domains.

2. PRELIMINARIES

Here in this section we recall some facts that will be used in our work. In all our work, Ω denotes a bounded and regular open set of $\mathbb{R}^n, n \geqslant 3$. For $q \in \mathbb{R}$ with $q \geqslant 2$, we denote by $L^q(\Omega) := \left\{u : \Omega \longrightarrow \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^q < \infty\right\}$. This set, equipped with the norm $\|u\|_{L^q(\Omega)} := \left(\int_\Omega |u|^q\right)^{1/q}$ for $u \in L^q(\Omega)$ is a Banach space. Note that, if $u \in L^q(\Omega)$, then $|u|^s \in L^{q/s}(\Omega)$ for all $1 \leqslant s \leqslant q$.

Proposition 2.1 (Holder's Inequality). *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, it holds that $\left|\int_\Omega fg\right| \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}$ for all $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$.*

Note that, Proposition 2.1 implies that (since Ω is bounded) $L^p(\Omega) \subset L^q(\Omega)$ for all $1 \leqslant q < p$. The Sobolev space is a very important set in studying PDEs. [Sobolev space] Let Ω be a bounded and regular open set of $\mathbb{R}^n, n \geqslant 2$. The Sobolev space $H^1(\Omega)$ is defined by $H^1(\Omega) := \left\{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega) \quad \forall\, i = 1, \cdots, n\right\}$, where $\frac{\partial u}{\partial x_i}$ denotes the derivative of u with respect to the i^{th} component of the variable x in the sense of distribution. In $H^1(\Omega)$, define

$$\langle u, v\rangle_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv, \quad \|u\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\right)^{1/2}.$$

Equipped with this scalar product, $H^1(\Omega)$ is a Hilbert space. Furthermore, we introduce $H_0^1(\Omega) := \overline{D(\Omega)}$, where $D(\Omega) := \{u \in C^\infty(\Omega) : \text{supp}(u) \text{ is a compact set of } \Omega\}$. Note that $H_0^1(\Omega)$ can be seen as $H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$. In addition, in $H_0^1(\Omega)$, the scalar product and its corresponding norm by:

$$\langle u, v\rangle_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v, \qquad \|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2\right)^{1/2}.$$

Equipped with this scalar product, $H_0^1(\Omega)$ is a Hilbert space. Furthermore, since Ω is bounded, the two norms $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent in $H_0^1(\Omega)$. Note that, from the definition, it is easy to deduce that $H_0^1(\Omega) \subset H^1(\Omega) \subset L^2(\Omega)$. However, we have a more general result. In fact, we have:

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Theorem 2.2 (Embedding Theorem). *Let $n \geqslant 3$ and Ω be a bounded and regular open set of $\mathbb{R}^n$. Then the embedding $\iota : H^1(\Omega) \hookrightarrow L^p(\Omega)$,*
(a) is compact for each $1 \leq p < \frac{2n}{n-2}$.
(b) is continuous for $p = \frac{2n}{n-2}$.

As a consequence of this theorem, we deduce that, for $n \geq 3$ and for each $1 \leq p \leq \frac{2n}{n-2}$, there exists a positive constant c such that $\|u\|_{L^p(\Omega)} \leq c\|u\|_{H_0^1(\Omega)}$ for all $u \in H_0^1(\Omega)$ and $\|u\|_{L^p(\Omega)} \leq c\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$.

Theorem 2.3 (Green's formula). *Let $u \in H^2(\Omega)$ and $v \in H_0^1(\Omega)$, it holds $\int_\Omega (-\Delta u)\, v = \int_\Omega \nabla u \cdot \nabla v$.*

Now we recall the Green's function. Let $n \geq 3$ and $\Gamma(\cdot,\cdot)$ be defined by $\Gamma(x,y) := \frac{1}{c_n|x-y|^{n-2}}$ for all $x, y \in \mathbb{R}^n$, $x \neq y$ where $c_n = n(n-2)\omega_n$ and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. The function $\Gamma(\cdot,\cdot)$ is called the normalized fundamental solution of Laplace's equation. For $x \in \mathbb{R}^n$ fixed, $\Gamma(x,\cdot)$ satisfies $-\Delta\Gamma(x,\cdot) = \delta_x$ in $\mathbb{R}^n$, where δ_x denotes the Dirac mass at the point x. When Ω is a bounded and regular open set of $\mathbb{R}^n$, $n \geq 3$, we introduce the Green's function $G(\cdot,\cdot)$ for the Laplace operator with Dirichlet boundary condition. This function satisfies, for $x \in \Omega$ fixed, $-\Delta G(x,\cdot) = c_n\delta_x$ in Ω and $G(x,\cdot) = 0$ on $\partial\Omega$. Written $G(\cdot,\cdot)$ as $G(x,y) := \frac{1}{|x-y|^{n-2}} - H(x,y)$, it follows that the function $H(\cdot,\cdot)$ satisfies, for $x \in \Omega$ fixed, $-\Delta H(x,\cdot) = 0$ in Ω and $H(x,y) = \frac{1}{|x-y|^{n-2}}$ on $\partial\Omega$ Note that $H\,(x,\cdot)$ is a harmonic function. To have some information about it, we need the following result extracted from the book of D. Gilbarg and N.S.T Rudiger.

Theorem 2.4 (Maximum and Minimal principle). *Let Ω be a bounded and regular open set of $\mathbb{R}^n$ and $u \in C^0(\overline{\Omega})$ be a harmonic function (that is $\Delta u = 0$ in Ω). Then*
(1) $\inf_{y\in\partial\Omega} u(y) \leqslant u(x) \leqslant \sup_{y\in\partial\Omega} u(y) \quad \forall\, x \in \Omega$,
(2) $|\nabla u(x)| \leqslant \dfrac{n}{d_x} \sup_{y\in\partial\Omega} |u(y)| \quad \forall\, x \in \Omega$, where $d_x := d(x, \partial\Omega)$.

Proposition 2.5 (Green's Representation formula). *Let Ω be a bounded and regular open set of $\mathbb{R}^n$, $n \geq 3$ and u be a function satisfying: $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$. Then it holds that $c_n u(y) = \int_\Omega G(x,y) f(x) dx$ for all $y \in \Omega$.*

Corollary 2.6. *Let $u \in H_0^1(\Omega)$ be a function satisfying: $-\Delta u \geq 0$ in Ω. Then it holds that $u > 0$ in Ω.*

3. ON APPROXIMATE SOLUTIONS

Note that when $\varepsilon \to 0$, the limit problem of (P_ε) can be seen as $-\Delta u = |u|^{4/(n-2)}u$ in $\mathbb{R}^n$. This problem is known as the Yamabe problem in $\mathbb{R}^n$ and it is very important to know the solutions of this Yamabe problem.

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Proposition 3.1. *Let $a \in \mathbb{R}^n$, $n \geqslant 3$, and $\lambda > 0$. The function $\delta_{a,\lambda}$, defined by Eq. (2), satisfies $-\Delta\delta_{a,\lambda} = \delta_{a,\lambda}^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$.*

Proof. Taking the derivative with respect to x_i, where $i \in \{1, 2, \ldots, n\}$, we obtain.

$$\frac{\partial \delta_{a,\lambda}}{\partial x_i} = c_0 \frac{-(n-2)\lambda^{(n+2)/2}(x_i - a_i)}{\left(1 + \lambda^2 |x-a|^2\right)^{\frac{n}{2}}}.$$

Therefore, the second derivative becomes

$$\frac{\partial^2 \delta_{a,\lambda}}{\partial x_i^2} = c_0 \frac{-(n-2)\lambda^{(n+2)/2}}{\left(1 + \lambda^2 |x-a|^2\right)^{\frac{n}{2}}} + c_0 \frac{n(n-2)\lambda^{(n+6)/2}(x_i - a_i)^2}{\left(1 + \lambda^2 |x-a|^2\right)^{(n+2)/2}}.$$

Thus we get

$$\begin{aligned}
\Delta\delta_{a,\lambda} = \sum_{i=1}^{n} \frac{\partial^2 \delta_{a,\lambda}}{\partial x_i^2} &= c_0 \frac{-n(n-2)\lambda^{(n+2)/2}}{\left(1 + \lambda^2 |x-a|^2\right)^{\frac{n}{2}}} + c_0 \frac{n(n-2)\lambda^{(n+6)/2}|x-a|^2}{\left(1 + \lambda^2 |x-a|^2\right)^{(n+2)/2}} \\
&= -n(n-2)c_0 \frac{\lambda^{(n+2)/2}}{\left(1 + \lambda^2 |x-a|^2\right)^{\frac{n+2}{2}}} \left(1 + \lambda^2|x-a|^2 - \lambda^2|x-a|^2\right) \\
&= -n(n-2)c_0 \frac{\lambda^{(n+2)/2}}{\left(1 + \lambda^2 |x-a|^2\right)^{\frac{n+2}{2}}} \\
&= -n(n-2)\frac{c_0}{c_0^{(n+2)/(n-2)}} \delta_{a,\lambda}^{(n+2)/(n-2)}.
\end{aligned}$$

Lemma 3.2. *Let $a \in \mathbb{R}^n$ and $\lambda > 0$. It holds that*

$$\lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} = \frac{n-2}{2} \delta_{a,\lambda} \frac{1 - \lambda^2|x-a|^2}{1 + \lambda^2|x-a|^2} \quad ; \qquad \left|\lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda}\right| \leq \frac{n-2}{2} \delta_{a,\lambda}.$$

Furthermore, for each $j \in \{1, \cdots, n\}$, we have

$$\frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j} = (n-2)\delta_{a,\lambda} \frac{\lambda(x_j - a_j)}{1 + \lambda^2|x-a|^2} \quad ; \qquad \left|\frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j}\right| \leq \frac{n-2}{2} \delta_{a,\lambda}.$$

Note that $\delta_{a,\lambda} > 0$ in $\mathbb{R}^n$ and therefore, for each $a \in \Omega$ and $\lambda > 0$, it follows that $\delta_{a,\lambda} \notin H_0^1(\Omega)$. For this reason, we will define the projection of $\delta_{a,\lambda}$ onto $H_0^1(\Omega)$. Let $P\delta_{a,\lambda}$ be this projection. Thus, $P\delta_{a,\lambda}$ satisfies

$$\begin{cases} -\Delta P\delta_{a,\lambda} = \delta_{a,\lambda}^{\frac{n+2}{n-2}} & \text{in } \Omega, \\ P\delta_{a,\lambda} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

In the sequel of this project, we will denote by $\theta_{a,\lambda} := \delta_{a,\lambda} - P\delta_{a,\lambda}$. Since $P\delta_{a,\lambda}$ is not given explicitly, we need a point-wise estimate. In fact, we have the following result which is very useful in the computations below.

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Proposition 3.3. *Let $a \in \Omega$ and $\lambda > 0$ be such that $\lambda d_a := \lambda d(a, \partial\Omega)$ is very large. It holds:*
(1) $0 \leqslant P\delta_{a,\lambda}(x) \leqslant \delta_{a,\lambda}(x) \quad \forall\, x \in \Omega,$
(2) $P\delta_{a,\lambda}(x) = \delta_{a,\lambda}(x) - \frac{c_0}{\lambda^{\frac{n-2}{2}}} H(a,x) + O\left(\frac{1}{\lambda^{\frac{n+2}{2}} d_a^n}\right) \quad \forall\, x \in \Omega,$
(3) $|\delta_{a,\lambda} - P\delta_{a,\lambda}|_{L^\infty(\Omega)} \leqslant \frac{C}{\lambda^{\frac{n-2}{2}} d_a^{n-2}}.$

Proof. (1) we know that $P\delta_{a,\lambda}$ satisfies (4). Using Proposition 2.5, we derive that

$$c_n P\delta_{a,\lambda}(y) = \int_\Omega G(x,y)\delta_{a,\lambda}^{(n+2)/(n-2)}(x)dx > 0,$$

which implies the first inequality in Assertion (1). Concerning the second inequality. It follows that

$$-\Delta\theta_{a,\lambda} = -\Delta\delta_{a,\lambda} - (-\Delta P\delta_{a,\lambda}) = \delta_{a,\lambda}^{(n+2)/(n-2)} - \left(\delta_{a,\lambda}^{(n+2)/(n-2)}\right) = 0.$$

Thus, $\theta_{a,\lambda}$ is a harmonic function. In addition, we have

$$\theta_{a,\lambda}(x) = \delta_{a,\lambda}(x) - P\delta_{a,\lambda}(x) = \delta_{a,\lambda}(x) > 0 \quad \text{for each} \quad x \in \partial\Omega.$$

Hence, using Theorem 2.4 (The minimal principal), it follows that

$$\theta_{a,\lambda}(x) \geq \inf_{y\in\partial\Omega} \theta_{a,\lambda}(y) > 0 \quad \forall\, x \in \Omega$$

which implies that $P\delta_{a,\lambda} < \delta_{a,\lambda}$ in Ω. This achieves the proof of Assertion (1).
(2) Concerning the second assertion, let

$$\phi(x) := \delta_{a,\lambda}(x) - \frac{c_0}{\lambda^{\frac{n-2}{2}}} H(a,x) - P\delta_{a,\lambda}(x) := \theta_{a,\lambda}(x) - \frac{c_0}{\lambda^{\frac{n-2}{2}}} H(a,x).$$

Since $\theta_{a,\lambda}$ and $H(a,.)$ are harmonic functions, we deduce that $\Delta\phi = 0$ in Ω. Therefore, to estimate ϕ, we need to evaluate it on the boundary. Let $x \in \partial\Omega$. Then

$$\begin{aligned}\phi(x) &= \delta_{a,\lambda}(x) - \frac{c_0}{\lambda^{(n-2)/2}} H(a,x) \\ &= c_0 \frac{\lambda^{(n-2)/2}}{\left(1+\lambda^2|x-a|^2\right)^{(n-2)/2}} - \frac{c_0}{\lambda^{(n-2)/2}} \frac{1}{|x-a|^{n-2}}.\end{aligned} \tag{5}$$

Observe that

$$\begin{aligned}\frac{c_0\lambda^{(n-2)/2}}{\left(1+\lambda^2|x-a|^2\right)^{(n-2)/2}} &= \frac{c_0\lambda^{(n-2)/2}}{(\lambda|x-a|)^{n-2}\left(\frac{1}{\lambda^2|x-a|^2}+1\right)^{(n-2)/2}} \\ &= \frac{c_0}{\lambda^{(n-2)/2}|x-a|^{n-2}}\left(1+\frac{1}{\lambda^2|x-a|^2}\right)^{-(n-2)/2}.\end{aligned} \tag{6}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Recall that $d_a := d(a, \partial\Omega) := \inf_{y\in\partial\Omega} |a-y| \leqslant |a-x|$ for all $x \in \partial\Omega$. Thus, for $x \in \partial\Omega$, we deduce that $\lambda^2|a-x|^2 \geq (\lambda d_a)^2$. Since λd_a is large, it follows that $\frac{1}{\lambda^2|x-a|^2}$ is small and therefore $\frac{1}{\lambda^2|x-a|^2} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Now, for $\gamma \in \mathbb{R}$, we remark that the function $f(t) := (1+t)^\gamma$, $\quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ is a C^∞-function on $[-\frac{1}{2}, \frac{1}{2}]$ and therefore, by using the Mean Value Theorem, we deduce that $|f(t)-f(0)| \leq \|f'\|_\infty|t| \leq c|t|$ for all $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Hence, we obtain for $\gamma \in \mathbb{R}$, $\quad (1+t)^\gamma = 1 + O(t)$ for all $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Hence

$$\frac{c_0\lambda^{(n-2)/2}}{\left(1+\lambda^2|x-a|^2\right)^{(n-2)/2}} = \frac{c_0}{\lambda^{\frac{n-2}{2}}|x-a|^{n-2}}\left\{1+O\left(\frac{1}{\lambda^2|x-a|^2}\right)\right\}$$
$$= \frac{c_0}{\lambda^{(n-2)/2}|x-a|^{n-2}} + O\left(\frac{1}{\lambda^{(n+2)/2}|x-a|^n}\right).$$

and therefore, by using (5), we derive that

$$|\phi(x)| \leq \frac{c}{\lambda^{\frac{n+2}{2}}|x-a|^n} \leq \frac{c}{\lambda^{\frac{n+2}{2}}d_a^n} \quad \forall\, x \in \partial\Omega.$$

Finally, since ϕ is a harmonic function, applying Theorem 2.4, we deduce that

$$-\frac{c}{\lambda^{\frac{n+2}{2}}d_a^n} \leqslant \min_{x\in\partial\Omega}\phi(y) \leqslant \phi(x) \leqslant \max_{x\in\partial\Omega}\phi(x) \leqslant \frac{c}{\lambda^{\frac{n+2}{2}}d_a^n} \qquad \forall\, y \in \Omega,$$

which implies the result and therefore the proof of Assertion (2) is completed.
(3) It remains to prove Assertion (3). from Assertion (2), we deduce that, for each $x \in \Omega$,

$$|\delta_{a,\lambda}(x) - P\delta_{a,\lambda}(x)| = \left|\frac{c_0}{\lambda^{\frac{n-2}{2}}}H(a,x) + O\left(\frac{1}{\lambda^{\frac{n+2}{2}}d_a^n}\right)\right|$$
$$\leqslant \frac{c_0}{\lambda^{\frac{n-2}{2}}}H(a,x) + \left|O\left(\frac{1}{\lambda^{\frac{n+2}{2}}d_a^n}\right)\right|.$$

Now, we obtain

$$|\delta_{a,\lambda}(x) - P\delta_{a,\lambda}(x)| \leqslant \frac{c_0}{\lambda^{\frac{n-2}{2}}}\frac{1}{d_a^{n-2}} + \frac{c}{\lambda^{\frac{n+2}{2}}d_a^n}$$
$$\leqslant \frac{c_0}{\lambda^{\frac{n-2}{2}}d_a^{n-2}}\left(1 + \frac{c}{(\lambda d_a)^2}\right)$$
$$\leqslant \frac{c}{\lambda^{\frac{n-2}{2}}d_a^{n-2}},$$

which completes the proof of Assertion (3). □

For the point-wise estimate of the derivative of the approximate solution, we have the following results:

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Proposition 3.4. *Let $a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large. It holds:*

(1) $\lambda \dfrac{\partial P\delta_{a,\lambda}}{\partial \lambda} = \lambda \dfrac{\partial \delta_{a,\lambda}}{\partial \lambda} + \dfrac{(n-2)}{2} c_0 \dfrac{H(a,.)}{\lambda^{(n-2)/2}} + O\left(\dfrac{1}{\lambda^{(n+2)/2} d_a^n}\right)$ *in* Ω,

(2) $\left|\lambda \dfrac{\partial P\delta_{a,\lambda}}{\partial \lambda}\right| \leqslant \dfrac{n+2}{2} P\delta_{a,\lambda} \leqslant \dfrac{n+2}{2} \delta_{a,\lambda}$ *in* Ω,

(3) $\left|\lambda \dfrac{\partial \theta_{a,\lambda}}{\partial \lambda}\right| \leqslant \dfrac{n-2}{2} \theta_{a,\lambda} \leqslant \dfrac{n-2}{2} \delta_{a,\lambda}$ *in* Ω.

Proof. (1) Note that, from Eq. (4), we deduce that

$$\begin{cases} -\Delta \left(\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\right) = \frac{n+2}{n-2} \delta_{a,\lambda}^{\frac{4}{n-2}} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} & \text{in } \Omega, \\ \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Now, let $\psi := \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} + \frac{(n-2)}{2} c_0 \frac{H(a,.)}{\lambda^{(n-2)/2}} - \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}$. By using (7), it follows that $\Delta\psi = 0$ in Ω Hence, using Theorem 2.4, we derive that $\inf_{y\in\partial\Omega} \psi(y) \leqslant \psi(x) \leqslant \sup_{y\in\partial\Omega} \psi(y)$ for all $x \in \Omega$ which implies that $|\psi(x)| \leqslant \sup_{y\in\partial\Omega} |\psi(y)|$ for all $x \in \Omega$. For $y \in \partial\Omega$, we have $|y-a| \geqslant d_a$ and therefore $\lambda|y-a|$ is very large for each $y \in \partial\Omega$. Thus, we obtain

$$\begin{aligned} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda}(y) &= c_0 \frac{(n-2)}{2} \lambda^{\frac{n-2}{2}} \frac{1 - \lambda^2|y-a|^2}{(1+\lambda^2|y-a|^2)^{n/2}} \\ &= c_0 \left(\frac{n-2}{2}\right) \lambda^{\frac{n-2}{2}} \left(\frac{-\lambda^2|y-a|^2}{(\lambda|y-a|)^n}\right) \left(1 - \frac{1}{\lambda^2|y-a|^2}\right) \left(1 + \frac{1}{\lambda^2|y-a|^2}\right)^{-\frac{n}{2}}. \end{aligned} \tag{8}$$

Since $\lambda|y-a|$ is large, using (3), we obtain:

$$\begin{aligned} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda}(y) &= \frac{-(n-2)}{2\lambda^{(n-2)/2}} \frac{c_0}{|y-a|^{n-2}} + O\left(\frac{1}{\lambda^{(n+2)/2}|y-a|^n}\right) \\ &= -c_0 \frac{(n-2)}{2\lambda^{(n-2)/2}} H(a,y) + O\left(\frac{1}{\lambda^{(n+2)/2} d_a^n}\right), \end{aligned}$$

which implies that, $|\psi(y)| \leqslant \frac{c}{\lambda^{(n+2)/2} d_a^n}$ for all $y \in \partial\Omega$.

(2) Using Proposition 2.5, we get, for each $y \in \Omega$,

$$c_n \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}(y) = \int_\Omega G(x,y)(-\Delta)\left(\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\right)(x)dx.$$

Or, using (7) and (8), we get

$$\left|-\Delta\left(\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\right)\right| = \frac{n+2}{n-2} \delta_{a,\lambda}^{\frac{4}{n-2}} \left|\lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda}\right| \leqslant \frac{n+2}{2} \delta_{a,\lambda}^{\frac{n+2}{n-2}}.$$

Thus, we obtain

$$c_n \left|\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}(y)\right| \leqslant \frac{n+2}{2} \int_\Omega G(x,y) \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)dx = \frac{n+2}{2} c_n P\delta_{a,\lambda}(y). \tag{9}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

(3) let $\psi_1 := \frac{n-2}{2}\theta_{a,\lambda} - \lambda\frac{\partial\theta_{a,\lambda}}{\partial\lambda}$ and $\psi_2 := \frac{n-2}{2}\theta_{a,\lambda} + \lambda\frac{\partial\theta_{a,\lambda}}{\partial\lambda}$. These functions are harmonic and they satisfy, for $y \in \partial\Omega$,

$$\psi_1(y) = \frac{n-2}{2}\delta_{a,\lambda}(y) - \lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}(y) \geq 0 \quad \text{and} \quad \psi_2(y) = \frac{n-2}{2}\delta_{a,\lambda}(y) + \lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}(y) \geq 0.$$

This ends the proof of the proposition. □

Proposition 3.5. *Let $a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large. For each $j \in \{1,\dots,n\}$, it holds,*

(1) $\dfrac{1}{\lambda}\dfrac{\partial P\delta_{a,\lambda}}{\partial a_j} = \dfrac{1}{\lambda}\dfrac{\partial\delta_{a,\lambda}}{\partial a_j} - -\dfrac{c_0}{\lambda^{n/2}}\dfrac{\partial H}{\partial a_j}(a,.) + O\left(\dfrac{1}{\lambda^{(n+4)/2}d_a^{n+1}}\right)$,

(2) $\left|\dfrac{1}{\lambda}\dfrac{\partial P\delta_{a,\lambda}}{\partial a_j}\right| \leqslant cP\delta_{a,\lambda}$.

Proof. From equation (4), we deduce that

$$\begin{cases} -\Delta\left(\dfrac{1}{\lambda}\dfrac{\partial P\delta_{a,\lambda}}{\partial a_j}\right) = \dfrac{n+2}{n-2}\delta_{a,\lambda}^{\frac{4}{n-2}}\dfrac{1}{\lambda}\dfrac{\partial\delta_{a,\lambda}}{\partial a_j} & \text{in} \quad \Omega, \\ \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{10}$$

Now, let $\psi_2 := \frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j} - \frac{c_0}{\lambda^{n/2}}\frac{\partial H}{\partial a_j}(a,\cdot) - \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}$ Then $\Delta\psi_2 = 0$ in Ω. Hence, using Theorem 2.4, we derive that $|\psi_2(x)| \leqslant \sup_{y\in\partial\Omega}|\psi_2(y)| \quad \forall x \in \Omega$. Now, for $y \in \partial\Omega$, we have $|y-a| \geqslant d_a$ and therefore $\lambda|y-a|$ is very large for each $y \in \partial\Omega$. Thus we obtain

$$\begin{aligned} \frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j}(y) &= \frac{1}{\lambda}(n-2)c_0\frac{\lambda^{(n+2)/2}(y_j-a_j)}{(1+\lambda^2|y-a|^2)^{n/2}} \\ &= (n-2)\frac{c_0}{\lambda^{n/2}}\frac{(y_j-a_j)}{|y-a|^n}\left(1+\frac{1}{\lambda^2|y-a|^2}\right)^{-n/2} \\ &= (n-2)\frac{c_0}{\lambda^{n/2}}\frac{y_j-a_j}{|y-a|^n} + O\left(\frac{1}{\lambda^{(n+2)/2}d_a^{n+1}}\right), \end{aligned} \tag{11}$$

by using Eq. (3) since $\lambda|y-a|$ is very large. Observe that,then

$$\frac{\partial H}{\partial a_j}(a,y) = (n-2)\frac{(y_j-a_j)}{|y-a|^n}.$$

Thus Eq. (11) becomes

$$\frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j}(y) = \frac{c_0}{\lambda^{n/2}}\frac{\partial H(a,y)}{\partial a_j} + O\left(\frac{1}{\lambda^{(n+2)/2}d_a^{n+1}}\right),$$

which implies that

$$|\psi_2(y)| \leqslant \frac{c}{\lambda^{(n+2)/2}d_a^{n+1}} \quad \forall y \in \partial\Omega. \tag{12}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Concerning Assertion (2), using Proposition 2.5 and Eq. (10), we deduce that

$$c_n \frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a_j}(y) = \int_\Omega G(x,y) \left(-\Delta \left(\frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a_j} \right) \right) (x) dx. \tag{13}$$

Note that

$$\begin{aligned} \left| -\Delta \left(\frac{1}{\lambda} \frac{\partial P\delta_{a,\lambda}}{\partial a_j} \right) \right| &= \frac{n+2}{n-2} \delta_{a,\lambda}^{\frac{4}{n-2}} \frac{1}{\lambda} \left| \frac{\partial \delta_{a,\lambda}}{\partial a_j} \right| \\ &= \frac{n+2}{n-2} \delta_{a,\lambda}^{\frac{4}{n-2}} c_0(n-2) \frac{\lambda^{n/2} |x_j - a_j|}{(1+\lambda^2|x-a|^2)^{n/2}} \\ &\leqslant \frac{n+2}{2} \delta_{a,\lambda}^{\frac{n+2}{n-2}}, \end{aligned}$$

by using the fact that $\frac{|t|}{1+t^2} \leqslant 1/2$ for all $t \in \mathbb{R}$. Since $G(x,y) > 0$, Eq. (13) implies that

$$c_n \frac{1}{\lambda} \left| \frac{\partial P\delta_{a,\lambda}}{\partial a_j} \right| (y) \leqslant \frac{n+2}{2} \int_\Omega G(x,y) \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x) dx = \frac{n+2}{2} c_n P\delta_{a,\lambda}(y).$$

Thus the proof of Proposition 3.5 is completed. □

4. MAIN RESULTS

In this section we present our main results which concern the estimating some integrals involving the function $P\delta_{a,\lambda}$ and its derivatives with respect to λ and the point a.

Theorem 4.1. *Let $n \geqslant 3$, $a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large. It holds*

(a) $\displaystyle\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}} = S + O\left(\frac{1}{(\lambda d_a)^n} \right)$ *where* $\displaystyle S = \int_{\mathbb{R}^n} \delta_{0,1}^{\frac{2n}{n-2}}$,

(b) $\displaystyle\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \theta_{a,\lambda} = O\left(\frac{1}{(\lambda d_a)^{n-2}} \right)$, *where* $\theta_{a,\lambda} = \delta_{a,\lambda} - P\delta_{a,\lambda}$,

(c) $\displaystyle\int_\Omega P\delta_{a,\lambda}^{\frac{2n}{n-2}} = S + O\left(\frac{1}{(\lambda d_a)^{n-2}} \right)$.

Proof. (a) We have

$$\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}} = \int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{2n}{n-2}} - \int_{\mathbb{R}^n \setminus \Omega} \delta_{a,\lambda}^{\frac{2n}{n-2}}. \tag{14}$$

From the definition of $\delta_{a,\lambda}$, see Eq. (2), it follows that

$$\int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{2n}{n-2}} = c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{\lambda^n}{(1+\lambda^2|x-a|^2)^n} dx.$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Using the change of variables $y = \lambda(x-a)$, we obtain

$$\int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{2n}{n-2}} = c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} dy = \int_{\mathbb{R}^n} \delta_{0,1}^{\frac{2n}{n-2}} = S. \tag{15}$$

For the other integral in Eq. (14), from the definition of $d_a := d(a, \partial\Omega) = \inf\{|x - a|,\ x \in \partial\Omega\}$, we deduce that $B(a,d) \subset \Omega$ and therefore $\mathbb{R}^n \backslash \Omega \subset \mathbb{R}^n \backslash B(a, d_a)$. Hence we obtain

$$\begin{aligned}\int_{\mathbb{R}^n\backslash\Omega} \delta_{a,\lambda}^{\frac{2n}{n-2}} &\leqslant \int_{\mathbb{R}^n\backslash B(a,d_a)} \delta_{a,\lambda}^{\frac{2n}{n-2}} = c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n\backslash B(a,d_a)} \frac{\lambda^n}{(1+\lambda^2|x-a|^2)^n} dx \\ &\leqslant \frac{C}{\lambda^n} \int_{\mathbb{R}^n\backslash B(a,d_a)} \frac{dx}{|x-a|^{2n}}.\end{aligned}$$

Passing to the polar coordinates, we obtain

$$\int_{\mathbb{R}^n\backslash B(a,d_a)} \delta_{a,\lambda}^{\frac{2n}{n-2}} \leqslant \frac{c}{\lambda^n} \int_{d_a}^{+\infty} \frac{r^{n-1}}{r^{2n}} dr \leqslant \frac{c}{\lambda^n} \int_{d_a}^{+\infty} \frac{dr}{r^{n+1}} \leqslant \frac{c}{(\lambda d_a)^n}. \tag{16}$$

(b) Using Proposition 3.3 and Theorem 4.4, we derive that

$$\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \theta_{a,\lambda} \leqslant \frac{c}{\lambda^{\frac{n-2}{2}} d_a^{n-2}} \int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \leqslant \frac{c}{\lambda^{\frac{n-2}{2}} d_a^{n-2}} \frac{c}{\lambda^{(n-2)/2}} = \frac{c}{(\lambda d_a)^{n-2}}$$

which completes the proof of Claim (b). It remains to prove Claim (c). Observe that, from Proposition 3.3, we deduce that $0 \leqslant \theta_{a,\lambda} \leqslant \delta_{a,\lambda}$. Obtain

$$P\delta_{a,\lambda}^{\frac{2n}{n-2}} = (\delta_{a,\lambda} - \theta_{a,\lambda})^{\frac{2n}{n-2}} = \delta_{a,\lambda}^{\frac{2n}{n-2}} + O\left(\delta_{a,\lambda}^{\frac{n+2}{n-2}} \theta_{a,\lambda}\right).$$

Hence, we get

$$\int_\Omega P\delta_{a,\lambda}^{\frac{2n}{n-2}} = \int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}} + O\left(\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \theta_{a,\lambda}\right).$$

The proof of Claim (c) follows by applying Claims (a) and (b). □

Theorem 4.2. *Let $n \geqslant 3$, $a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large.*
(a) For each $v \in H_0^1(\Omega)$, it holds

$$\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}} \theta_{a,\lambda} |v| \leqslant c\|v\| \times \begin{cases} (\lambda d_a)^{-(n-2)} & \text{if} \quad n \leqslant 5, \\ \ln^{2/3}(\lambda d_a)\,(\lambda d_a)^{-4} & \text{if} \quad n = 6, \\ (\lambda d_a)^{-(n+2)/2} & \text{if} \quad n \geqslant 7. \end{cases}$$

(b) Let $v \in H_0^1(\Omega)$ be such that $\int_\Omega \nabla P\delta_{a,\lambda} \cdot \nabla v = 0$. Then, it holds

$$\left|\int_\Omega P\delta_{a,\lambda}^{\frac{n+2}{n-2}} v\right| \leqslant c\|v\| \times \begin{cases} (\lambda d_a)^{-(n-2)} & \text{if} \quad n \leqslant 5, \\ \ln^{2/3}(\lambda d_a)\,(\lambda d_a)^{-4} & \text{if} \quad n = 6, \\ (\lambda d_a)^{-(n+2)/2} & \text{if} \quad n \geqslant 7. \end{cases}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Proof. By using the Holder's Inequality (see Proposition 2.1), we derive that

$$\int_{\Omega} \delta_{a,\lambda}^{\frac{4}{n-2}} \theta_{a,\lambda}|v| \leqslant \left(\int_{\Omega} |v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \left(\int_{\Omega} \left(\delta_{a,\lambda}^{\frac{4}{n-2}} \theta_{a,\lambda}\right)^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}}. \tag{17}$$

Now, we have

$$\left(\int_{\Omega} |v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} = \|v\|_{L^{2n/(n-2)}(\Omega)} \leqslant c\|v\|. \tag{18}$$

For the other integral in Eq. (17), using Proposition 3.3, we get,

$$\begin{aligned} \int_{B(a,d_a)} \delta_{a,\lambda}^{\frac{8n}{(n-2)(n+2)}} \theta_{a,\lambda}^{\frac{2n}{n+2}} &\leqslant \frac{c}{(\lambda d_a^2)^{\frac{n(n-2)}{n+2}}} \int_{B(a,d_a)} \frac{\lambda^{\frac{4n}{n+2}}}{(1+\lambda^2|x-a|^2)^{\frac{4n}{n+2}}} dx \\ &\leqslant \frac{c}{(\lambda d_a)^{2\frac{n(n-2)}{n+2}}} \int_0^{\lambda d_a} \frac{r^{n-1}}{(1+r^2)^{\frac{4n}{n+2}}} dr, \end{aligned} \tag{19}$$

by using the change of variables $y = \lambda(x-a)$ and polar coordinates. If $n = 6$, the integral in Eq. (19) gives

$$\begin{aligned} \int_0^{\lambda d_a} \frac{r^5}{(1+r^2)^3} dr &\leq \int_0^1 \frac{r^5}{(1+r^2)^3} dr + \int_1^{\lambda d_a} \frac{dr}{r} \\ &\leqslant c + \ln(\lambda d_a) \leqslant 2\ln(\lambda d_a), \end{aligned}$$

since λd_a is large. If if $n \leqslant 5$, the last integral in (19) is bounded, indeed,

$$\int_0^{\lambda d_a} \frac{r^{n-1}}{(1+r^2)^{4n/(n+2)}} dr \leqslant \int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^{4n/(n+2)}} dr \leqslant c,$$

since $\frac{8n}{n+2} - n + 1 = \frac{n}{n+2}(6-n) + 1 > 1$ for $n \leqslant 5$. If $n \geqslant 7$, it holds

$$\int_0^{\lambda d_a} \frac{r^{n-1}}{(1+r^2)^{4n/(n+2)}} dr \leqslant \int_0^{\lambda d_a} r^{n-\frac{8n}{n+2}-1} dr \leqslant c\,(\lambda d_a)^{n-\frac{8n}{n+2}},$$

since $n - \dfrac{8n}{n+2} = \dfrac{n}{n+2}(n-6) > 0$ for $n \geqslant 7$. Thus, combining these estimates with Eq. (19), we obtain

$$\int_{B(a,d_a)} \delta_{a,\lambda}^{\frac{8n}{(n-2)(n+2)}} \theta_{a,\lambda}^{\frac{2n}{n+2}} \leqslant c \times \begin{cases} \ln(\lambda d_a) / (\lambda d_a)^6 & \text{if } n = 6, \\ 1/(\lambda d_a)^{2n(n-2)/(n+2)} & \text{if } n \leqslant 5, \\ 1/(\lambda d_a)^n & \text{if } n \geqslant 7. \end{cases} \tag{20}$$

It remains to estimate the integral over $\Omega \backslash B(a, d_a)$. Using Proposition 3.3, we know that $\theta_{a,\lambda} \leqslant \delta_{a,\lambda}$ and therefore we get, by using Eq. (16),

$$\int_{\Omega\backslash B(a,d_a)} \delta_{a,\lambda}^{\frac{8n}{(n-2)(n+2)}} \theta_{a,\lambda}^{\frac{2n}{n+2}} \leqslant \int_{\Omega\backslash B(a,d_a)} \delta_{a,\lambda}^{\frac{2n}{n-2}} \leqslant \frac{c}{(\lambda d_a)^n}. \tag{21}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Combining Eqs. (20) and (21), we derive that

$$\left(\int_\Omega \delta_{a,\lambda}^{\frac{8n}{(n-2)(n+2)}} \theta_{a,\lambda}^{\frac{2n}{n-2}}\right)^{\frac{n+2}{2n}} \leqslant c \times \begin{cases} (\lambda d_a)^{-(n-2)} & \text{if} \quad n \leqslant 5, \\ \ln^{2/3}(\lambda d_a)(\lambda d_a)^{-4} & \text{if} \quad n = 6, \\ (\lambda d_a)^{-(n+2)/2} & \text{if} \quad n \geqslant 7. \end{cases} \tag{22}$$

Thus, the proof of Assertion (a) follows from Eqs. (17), (18) and (22). Concerning (b), since $\theta_{a,\lambda} \leq \delta_{a,\lambda}$, we derive that

$$\int_\Omega P\delta_{a,\lambda}^{\frac{n+2}{n-2}} v = \int_\Omega (\delta_{a,\lambda} - \theta_{a,\lambda})^{\frac{n+2}{n-2}} v = \int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} v + O\left(\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}} \theta_{a,\lambda}|v|\right).$$

The last integral is computed in Assertion (a). For the first one, using Theorem 2.3, it follows that

$$\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} v = \int_\Omega (-\Delta P\delta_{a,\lambda}) v = \int_\Omega \nu P\delta_{a,\lambda} \nu v = 0.$$

This ends the proof. □

Theorem 4.3. *Let* $n \geqslant 3$ *and* $v \in H_0^1(\Omega)$. *It holds*

(a) $\displaystyle\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}-\beta} |v|^\beta \leqslant c\|v\|^\beta \quad \forall \beta \in \left(0, \frac{2n}{n-2}\right)$,

(b) $\displaystyle\int_\Omega \delta_{a,\lambda}|v| \leqslant c\|v\| \times \begin{cases} \lambda^{-(n-2)/2} & \textit{if} \quad n \leqslant 5, \\ (\ln \lambda)^{2/3} \lambda^{-2} & \textit{if} \quad n = 6, \\ \lambda^{-2} & \textit{if} \quad n \geqslant 7. \end{cases}$

Proof. (a) Using the Holder's Inequality (see Proposition 2.1), Claim (a) of Theorem 4.1, we obtain

$$\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}-\beta} |v|^\beta \leqslant \left(\int_\Omega |v|^{\frac{2n}{n-2}}\right)^{\frac{\beta(n-2)}{2n}} \left(\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}}\right)^{\frac{2n-\beta(n-2)}{2n}} \leqslant c\|v\|^\beta.$$

This ends the proof of Claim (a).
(b) Using again the Holder's Inequality (see Proposition 2.1), we get

$$\int_\Omega \delta_{a,\lambda}|v| \leqslant \left(\int_\Omega |v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \left(\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}}. \tag{23}$$

Now, we need to estimate the last integral in (23). In fact, it holds. If $n \geqslant 7$, using the definition of $\delta_{a,\lambda}$ (see Eq. (2)), the change of variables $y = \lambda(x - a)$ and the polar

*Corresponding Author: Ayidh Munahi Bati Alqahtani

coordinates, we get

$$\begin{aligned}\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n+2}} &\leqslant c\int_{\mathbb{R}^n} \frac{\lambda^{n(n-2)/(n+2)}}{(1+\lambda^2|x-2|^2)^{n(n-2)/(n+2)}}dx \\ &\leqslant \frac{c\lambda^{n(n-2)/(n+2)}}{\lambda^n}\int_{\mathbb{R}^n}\frac{1}{(1+|y|^2)^{n(n-2)/(n+2)}}dy \\ &\leqslant \frac{c}{\lambda^{4n/(n+2)}}\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n(n-2)/(n+2)}}dr \\ &\leqslant \frac{c}{\lambda^{4n/(n+2)}}, \end{aligned} \tag{24}$$

since $2n\frac{n-2}{n+2} - n + 1 = n\frac{n-6}{n-2} + 1 > 1$ for $n \geqslant 7$ and therefore the last integral is convergent. If $n = 6$, then $\frac{2n}{n+2} = \frac{3}{2}$. Let $R > 0$ be such that $\Omega \subset B(a, R)$ (since Ω is bounded). Using again the change of variables $y = \lambda(x - a)$ and the polar coordinates, we get

$$\begin{aligned}\int_\Omega \delta^{\frac{2n}{n+2}} &\leqslant \int_{B(a,R)} \delta_{a,\lambda}^{3/2} \leqslant c\int_{B(a,R)}\frac{\lambda^3}{(1+\lambda^2|x-a|^2)^3}dx \\ &\leqslant \frac{c}{\lambda^6}\int_{B(0,\lambda R)}\frac{\lambda^3}{(1+|y|^2)^3}dy \\ &\leqslant \frac{c}{\lambda^3}\int_0^{\lambda R}\frac{r^5}{(1+r^2)^3}dr \\ &\leqslant \frac{c}{\lambda^3}\left(\int_0^1\frac{r^5}{(1+r^2)^3}dr + \int_1^{\lambda R}\frac{dr}{r}\right) \\ &\leqslant \frac{c}{\lambda^3}(c+\ln(\lambda R)) = \frac{c}{\lambda^3}(c+\ln\lambda+\ln R) \\ &\leqslant c\frac{\ln\lambda}{\lambda^3}, \end{aligned} \tag{25}$$

since $\ln\lambda$ is large. If $n \leqslant 5$,

$$\begin{aligned}\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n+2}} &\leqslant c\int_{B(a,R)}\frac{\lambda^{n(n-2)/(n+2)}}{(1+\lambda^2|x-a|^2)^{n(n-2)/(n+2)}}dx \\ &\leqslant \frac{c}{\lambda^{n(n-2)/(n+2)}}\int_{B(a,R)}\frac{dx}{|x-a|^{2n(n-2)/(n+2)}} \\ &\leqslant \frac{c}{\lambda^{n(n-2)/(n+2)}}\int_{B(0,R)}\frac{dy}{|y|^{2n(n-2)/(n+2)}} \\ &\leqslant \frac{c}{\lambda^{n(n-2)/(n+2)}}\int_0^R r^{n-1-2n(n-2)/(n+2)}dr \\ &\leqslant \frac{c}{\lambda^{n(n-2)/(n+2)}}, \end{aligned} \tag{26}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

since $n - 1 - 2n\frac{(n-2)}{n+2} = n\frac{6-n}{n+2} - 1 > -1$ for $n \leqslant 5$ and therefore the last integral is convergent. Combining Eqs. (23)-(26), the proof of theorem follows. □

Theorem 4.4. *Let $n \geqslant 4$, $a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large. It holds*

(a) $\displaystyle\int_\Omega \delta_{a,\lambda}^2 \leqslant \begin{cases} c(\ln\lambda)\lambda^{-2} & \text{if } n = 4, \\ c\lambda^{-2} & \text{if } n \geqslant 5, \end{cases}$

(b) $\displaystyle\int_{B(a,d_a)} \delta_{a,\lambda}^{\frac{n}{n-2}} \leq c\frac{\ln(\lambda d_a)}{\lambda^{n/2}},$

(c) $\displaystyle\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} = \frac{\bar{c}_1}{\lambda^{(n-2)/2}} + O\left(\frac{1}{\lambda^{\frac{n+2}{2}} d_a^2}\right),$

(d) $\displaystyle\int_\Omega \delta_{a,\lambda} \leq \frac{c}{\lambda^{(n-2)/2}}$, *where c_0 is defined in Eq. (2) and*

$$\bar{c}_1 := c_0^{\frac{n+2}{n-2}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+2)/2}}.$$

Proof. (a) Note that, since Ω is bounded, there exists $R > 0$ such that $\Omega \subset B(a,R)$. Using the definition of $\delta_{a,\lambda}$ given in Eq. (2), the change of variables $y = \lambda(x-a)$ and the polar coordinates, we get

$$\begin{aligned}\int_\Omega \delta_{a,\lambda}^2 &\leqslant c\int_{B(a,R)} \frac{\lambda^{n-2}}{(1+\lambda^2|x-a|^2)^{n-2}}dx \\ &\leqslant \frac{c}{\lambda^n}\int_{B(0,\lambda R)} \frac{\lambda^{n-2}}{(1+|y|^2)^{n-2}}dy \\ &\leqslant \frac{c}{\lambda^2}\int_0^{\lambda R} \frac{r^{n-1}}{(1+r^2)^{n-2}}dr.\end{aligned}$$

Observe that, if $n \geqslant 5$, then $2(n-2)-n+1 = n-3 > 1$ and therefore $\int_0^{\lambda R} \frac{r^{n-1}}{(1+r^2)^{n-2}}dr \leqslant \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n-2}}dr \leqslant c$. But, if $n = 4$, the integral becomes

$$\begin{aligned}\int_0^{\lambda R} \frac{r^{n-1}}{(1+r^2)^{n-2}}dr = \int_0^{\lambda R} \frac{r^3}{(1+r^2)^2}dr &\leqslant \int_0^1 \frac{r^3}{(1+r^2)^2}dr + \int_1^{\lambda R} \frac{dr}{r} \\ &\leqslant c + \ln(\lambda R) \\ &\leqslant c + \ln\lambda + \ln R \\ &\leqslant c\ln\lambda,\end{aligned}$$

since $\ln\lambda$ is large. This achieves the proof of Assertion (a).

(b) We have

$$\int_{B(a,d_a)} \delta_{a,\lambda}^{\frac{n}{n-2}} \leqslant c\int_{B(a,d_a)} \frac{\lambda^{n/2}}{\left(1+\lambda^2\left|x-a\right|^2\right)^{n/2}}dx.$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Using the change of variables $y=\lambda(x-a)$ and the polar coordinates, we obtain

$$\begin{aligned}\int_{B(a,d_a)} \delta_{a,\lambda}^{\frac{n}{n-2}} &\leqslant \frac{c}{\lambda^{n/2}} \int_{B(0,\lambda d_a)} \frac{1}{(1+|y|^2)^{n/2}} dy \\ &\leqslant \frac{c}{\lambda^{n/2}} \int_0^{\lambda d_a} \frac{r^{n-1}}{(1+r^2)^{n/2}} dr \\ &\leqslant \frac{c}{\lambda^{n/2}} \left(\int_0^1 \frac{r^{n-1}}{(1+r^2)^{n/2}} dr + \int_1^{\lambda d_a} \frac{1}{r} dr \right) \\ &\leqslant \frac{c}{\lambda^{n/2}} \left(c+\ln(\lambda d_a)\right) \\ &\leqslant c\frac{\ln(\lambda d_a)}{\lambda^{n/2}}.\end{aligned}$$

(c) Note that

$$\int_{\Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} = \int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{n+2}{n-2}} - \int_{\mathbb{R}^n \setminus \Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}}. \tag{27}$$

The second one can be computed as

$$\begin{aligned}\int_{\mathbb{R}^n \setminus \Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} &\leqslant c \int_{\mathbb{R}^n \setminus B(a,d_a)} \frac{\lambda^{(n+2)/2}}{\left(1+\lambda^2 |x-a|^2\right)^{(n+2)/2}} \\ &\leqslant \frac{c}{\lambda^{(n+2)/2}} \int_{\mathbb{R}^n \setminus B(a,d_a)} \frac{1}{|x-a|^{n+2}} dx \\ &\leqslant \frac{c}{\lambda^{(n+2)/2}} \int_{d_a}^{\infty} \frac{1}{r^3} dr \\ &\leqslant \frac{c}{\lambda^{(n+2)/2} d_a^2}.\end{aligned} \tag{28}$$

For the first integral in Eq. (27), Using the change of variables $y=\lambda(x-a)$, we get

$$\int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{n+2}{n-2}} = c_0^{\frac{n+2}{n-2}} \int_{\mathbb{R}^n} \frac{\lambda^{(n+2)/2}}{\left(1+\lambda^2 |x-a|^2\right)^{\frac{n+2}{2}}} dx = c_0^{\frac{n+2}{n-2}} \frac{1}{\lambda^{\frac{n-2}{2}}} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{(n+2)/2}} dy.$$

This completes the proof of Theorem 4.4. □

Theorem 4.5. *Let $n \geqslant 4$, $a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large. It holds:*

(a) $\displaystyle\int_{\Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} = O\left(\frac{1}{(\lambda d_a)^n}\right)$,

(b) $\displaystyle\int_{\Omega} P\delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} = 2\left\langle P\delta_{a,\lambda}, \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\right\rangle + O\left(\frac{\ln(\lambda d_a)}{(\lambda d_a)^n}\right)$,

(c) $\displaystyle\left\langle P\delta_{a,\lambda}, \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\right\rangle = \frac{n-2}{2} \widetilde{c} \frac{H(a,a)}{\lambda^{n-2}} + O\left(\frac{\ln(\lambda d_a)}{(\lambda d_a)^n}\right)$, *where* $\widetilde{c} := c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+2}{2}}}$, *and c_0 is defined in Eq. (2).*

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Proof. (a) Note that, from (15), we know that

$$\int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{2n}{n-2}} = S, \quad \forall a \in \mathbb{R}^n, \quad \forall \lambda > 0,$$

which implies that

$$0 = \frac{\partial}{\partial \lambda} \int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{2n}{n-2}} = \frac{2n}{n-2} \int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{\partial \delta_{a,\lambda}}{\partial \lambda}. \tag{29}$$

Furthermore, using Theorem 3.2 and (16) we get

$$\left| \int_{\mathbb{R}^n \setminus \Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} \right| \leq c \int_{\mathbb{R}^n \setminus \Omega} \delta_{a,\lambda}^{\frac{2n}{n-2}} \leq c \int_{\mathbb{R}^n \setminus B(a,d_a)} \delta_{a,\lambda}^{\frac{2n}{n-2}} \leq \frac{c}{(\lambda d_a)^n}. \tag{30}$$

Thus, (29) and (30) imply that

$$\int_{\Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} = \int_{\mathbb{R}^n} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} - \int_{\mathbb{R}^n \setminus \Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} = O\left(\frac{1}{(\lambda d_a)^n} \right).$$

(b) Using Proposition 3.3, we have

$$0 \leqslant \theta_{a,\lambda} := \delta_{a,\lambda} - P\delta_{a,\lambda} \leqslant \delta_{a,\lambda} \quad \text{in} \ \ \Omega. \tag{31}$$

Hence it follows that

$$\begin{aligned} \int_{\Omega} P\delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} &= \int_{\Omega} (\delta_{a,\lambda} - \theta_{a,\lambda})^{\frac{n+2}{n-2}} \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} \\ &= \int_{\Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} - \frac{n+2}{n-2} \int_{\Omega} \delta_{a,\lambda}^{\frac{4}{n-2}} \theta_{a,\lambda} \left(\lambda \frac{\partial \delta_{a,\lambda}}{\partial \lambda} - \lambda \frac{\partial \theta_{a,\lambda}}{\partial \lambda} \right) \\ &\quad + O\left(\int_{\Omega} \delta_{a,\lambda}^{\frac{6-n}{n-2}} \theta_{a,\lambda}^2 \left| \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} \right| \right). \end{aligned} \tag{32}$$

For the first integral in Eq. (32), using Theorem 2.3 and the fact that $\frac{\partial P\delta_{a,\lambda}}{\partial \lambda} = 0$ on $\partial\Omega$, we derive that

$$\begin{aligned} \int_{\Omega} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} &= \int_{\Omega} (-\Delta P\delta_{a,\lambda}) \left(\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} \right) \\ &= \int_{\Omega} \nabla P\delta_{a,\lambda} \nabla \left(\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} \right) \\ &= \left\langle P\delta_{a,\lambda}, \lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda} \right\rangle. \end{aligned} \tag{33}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Concerning the second integral in Eq. (32), it will be divided into two pieces:

$$\begin{aligned}\frac{n+2}{n-2}\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}\theta_{a,\lambda} &= \int_{\Omega}\left(-\Delta\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right)(\delta_{a,\lambda}-P\delta_{a,\lambda})\\ &= \frac{n+2}{n-2}\int_{\Omega}\delta_{a,\lambda}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}-\int_{\Omega}\left(-\Delta\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right)P\delta_{a,\lambda}\\ &= O\left(\frac{1}{(\lambda d_a)^n}\right)-\int_{\Omega}\nabla\left(\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right)\nabla P\delta_{a,\lambda}\\ &= -\left\langle P\delta_{a,\lambda},\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right\rangle+O\left(\frac{1}{(\lambda d_a)^n}\right),\end{aligned}\tag{34}$$

(c) Using Propositions 3.3 and 3.4 we get (since $n\geqslant 4$)

$$\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}\left|\lambda\frac{\partial\theta_{a,\lambda}}{\partial\lambda}\right|\leqslant\frac{c}{(\lambda d_a^2)^{n/2}}\int_{B(a,d_a)}\delta_{a,\lambda}^{\frac{n}{n-2}}+c\int_{\Omega\setminus B(a,d_a)}\delta_{a,\lambda}^{\frac{2n}{n-2}}\leqslant c\frac{\ln(\lambda d_a)}{(\lambda d_a)^n}.\tag{35}$$

where we have used Assertion (b) of Theorem 4.4 and Eq. (16). It remains to estimate the last integral in Eq. (32). Using Propositions 3.4 and 3.3, we get:

$$\begin{aligned}\int_{\Omega}\delta_{a,\lambda}^{\frac{6-n}{n-2}}\theta_{a,\lambda}^2\left|\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right| &\leqslant c\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}^2\\ &\leqslant\frac{c}{(\lambda d_a^2)^{n/2}}\int_{B(a,d_a)}\delta_{a,\lambda}^{\frac{n}{n-2}}+c\int_{\Omega\setminus B(a,d_a)}\delta_{a,\lambda}^{\frac{2n}{n-2}}\\ &\leqslant c\frac{\ln(\lambda d_a)}{(\lambda d_a)^n},\end{aligned}\tag{36}$$

where we have used (16) and Assertion (b) of Theorem 4.4 in the last step. Combining Eq. (32)-(36), we obtain

$$\int_{\Omega}P\delta_{a,\lambda}^{\frac{n+2}{n-2}}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}=2\langle P\delta_{a,\lambda},\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\rangle+O\left(\frac{\ln(\lambda d_a)}{(\lambda d_a)^n}\right),$$

Lastly, we will focus on proving Assertion (c). Using Proposition 3.4, we have

$$\begin{aligned}<P\delta_{a,\lambda},\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}> &= \int_{\Omega}\nabla P\delta_{a,\lambda}\nabla\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\\ &= \int_{\Omega}(-\Delta P\delta_{a,\lambda})\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\\ &= \int_{\Omega}\delta_{a,\lambda}^{\frac{n+2}{n-2}}\left\{\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}+\frac{n-2}{2}c_0\frac{H(a,\cdot)}{\lambda^{(n-2)/2}}+O\left(\frac{1}{\lambda^{\frac{n+2}{2}}d_a^n}\right)\right\}\\ &= \int_{\Omega}\delta_{a,\lambda}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}+\frac{n-2}{2}\frac{c_0}{\lambda^{(n-2)/2}}\int_{\Omega}\delta_{a,\lambda}^{\frac{n+2}{n-2}}H(a,\cdot)\\ &\quad+O\left(\frac{1}{\lambda^{(n+2)/2}d_a^n}\int_{\Omega}\delta_{a,\lambda}^{\frac{n+2}{n-2}}\right).\end{aligned}\tag{37}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

The first integral is computed in Assertion (a). For the last one, using Assertion (c) of Theorem 4.4, we obtain

$$\frac{1}{\lambda^{(n+2)/2} d_a^n} \int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \leqslant \frac{1}{\lambda^{(n+2)/2} d_a^n} \frac{c}{\lambda^{(n-2)/2}} = \frac{c}{(\lambda d_a)^n}. \tag{38}$$

Concerning the second integral in Eq. (37), expanding $H(a, \cdot)$ around a and using the fact that $H(a, \cdot)$ is harmonic, using Theorem 2.4, we get, in $B(a, d_a/2)$,

$$H(a,x) = H(a,a) + \frac{\partial}{\partial b} H(a,a)(x-a) + O\left(\frac{|x-a|^2}{d_a^n}\right),$$

where $\partial H/\partial b$ denotes the derivative of H with respect the second variable. We notice that, in the last formula, we used the fact that: $d_a/2 \leqslant d_x \leqslant (3/2)d_a$ for each $x \in B(a, d_a/a)$.

Thus, we obtain

$$\begin{aligned} \int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x) H(a,x)dx = & \int_{B(a,d_a/2)} \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x) H(a,a)dx \\ & + \sum_{j=1}^n \frac{\partial H}{\partial b_j}(a,a) \int_{B(a,d_a/2)} (x_j - a_j)\, \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)dx \\ & + O\left(\frac{1}{d_a^n} \int_{B(a,d_a/2)} |x-a|^2\, \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)dx\right) \\ & + O\left(\int_{\Omega \setminus B(a,d_a/2)} \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x) H(a,x)dx\right). \end{aligned} \tag{39}$$

Using Theorem 4.4, we deduce that

$$\frac{H(a,a)}{\lambda^{\frac{n-2}{2}}} \int_{B(a,d_a/2)} \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)dx = \bar{c}_1 \frac{H(a,a)}{\lambda^{(n-2)}} + O\left(\frac{1}{(\lambda d_a)^n}\right). \tag{40}$$

For the second integral in Eq. (39), since the function is even with respect to $(x_j - a_j)$, we derive that

$$\int_{B(a,d_a/2)} (x_j - a_j)\, \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)dx = 0. \tag{41}$$

Now, for the third integral in Eq. (39), we have

$$\begin{aligned} \int_{B(a,d_a/2)} |x-a|^2\, \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)dx & \leqslant c \int_{B(a,d_a/2)} \frac{\lambda^{(n-2)/2} \lambda^2 |x-a|^2}{(1+\lambda^2 |x-a|^2)^{(n+2)/2}} dx \\ & \leqslant \frac{c}{\lambda^{(n+2)/2}} \int_0^{\lambda d_a/2} \frac{r^{n+1}}{(1+r^2)^{(n+2)/2}} \\ & \leqslant c \frac{\ln(\lambda d_a)}{\lambda^{(n+2)/2}}. \end{aligned} \tag{42}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Finally, for the last integral in Eq. (39), we get

$$\begin{aligned}\int_{\Omega\setminus B(a,d_a/2)} \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)H(a,x)dx &\leqslant \frac{c}{d_a^{n-2}}\int_{\Omega\setminus B(a,d_a/2)} \delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)dx \\ &\leqslant \frac{c}{d_a^{n-2}}\frac{1}{\lambda^{(n+2)/2}}\int_{\mathbb{R}^n\setminus B(a,d_a/2)}\frac{dx}{|x-a|^{n+2}} \\ &\leqslant \frac{c}{\lambda^{\frac{n+2}{2}}d_a^{n-2}}\int_{d_a/2}^{\infty}\frac{1}{r^3}dr \\ &\leqslant \frac{c}{\lambda^{\frac{n+2}{2}}d_a^{n}}. \end{aligned} \tag{43}$$

Using Eqs. (40) - (43), the equation (39) becomes

$$\frac{c_0}{\lambda^{\frac{n-2}{2}}}\int_{\Omega}\delta_{a,\lambda}^{\frac{n+2}{n-2}}(x)H(a,x)dx = c_0\bar{c}_1\frac{H(a,a)}{\lambda^{(n-2)}} + O\left(\frac{\ln(\lambda d_a)}{(\lambda d_a)^n}\right). \tag{44}$$

Combining Eq. (38), (44) and Assertion (a), the equation (37) becomes

$$\left\langle P\delta_{a,\lambda}, \lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right\rangle = \frac{n-2}{2}c_0\bar{c}_1\frac{H(a,a)}{\lambda^{n-2}} + O\left(\frac{\ln(\lambda d_a)}{(\lambda d_a)^n}\right).$$

This completes the proof. □

Theorem 4.6. *Let $a\in\Omega$ and $\lambda>0$ be such that λd_a is very large. Then, for $n\geqslant 5$, it holds:*

$$\int_\Omega P\delta_{a,\lambda}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda} = -\frac{c_2}{\lambda^2} + O\left(\frac{1}{(\lambda d_a)^{n-2}}\right),$$

with $c_2 := \dfrac{n-2}{2}c_0^2\displaystyle\int_{\mathbb{R}^n}\frac{|y|^2-1}{(1+|y|^2)^{n-1}}dy$. For $n=4$, assume that $d_a := d(a,\partial\Omega)\geqslant c>0$, then it holds:

$$\int_\Omega P\delta_{a,\lambda}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda} = -c_0^2\omega_3\frac{\ln\lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right),$$

where ω_3 is the area of the unit sphere of $\mathbb{R}^4$.

Proof. Observe that

$$\int_\Omega P\delta_{a,\lambda}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda} = \int_\Omega \delta_{a,\lambda}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda} - \int_\Omega\theta_{a,\lambda}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda} - \int_\Omega\delta_{a,\lambda}\lambda\frac{\partial\theta_{a,\lambda}}{\partial\lambda}.$$

Using Propositions 3.3, 3.4 and Theorem 4.4, we get

$$\begin{aligned}\left|\int_\Omega\theta_{a,\lambda}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right| + \left|\int_\Omega\delta_{a,\lambda}\lambda\frac{\partial\theta_{a,\lambda}}{\partial\lambda}\right| &\leqslant c\int_\Omega\theta_{a,\lambda}\delta_{a,\lambda} \\ &\leqslant \frac{c}{\lambda^{(n-2)/2}d^{n-2}}\int_\Omega\delta_{a,\lambda} \\ &\leqslant \frac{c}{\lambda^{n-2}d^{n-2}}\int_\Omega\frac{dx}{|x-a|^{n-2}} \\ &\leqslant \frac{c}{(\lambda d)^{n-2}}.\end{aligned}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

For the other integral, we have:

$$\begin{aligned}\int_\Omega \delta_{a,\lambda}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda} &= c_0^2\frac{n-2}{2}\int_\Omega \frac{\lambda^{n-2}(1-\lambda^2\left|x-a\right|^2)}{(1+\lambda^2|x-a|^2)^{n-1}}dx\\ &= \frac{n-2}{2}c_0^2\left(\int_{B(a,d)}\cdots+\int_{\Omega\setminus B(a,d)}\cdots\right)\\ &= \frac{n-2}{2}c_0^2\frac{1}{\lambda^2}\int_{B(0,\lambda d)}\frac{1-|y|^2}{(1+|y|^2)^{n-1}}dy + O\left(\frac{1}{\lambda^{n-2}}\int_{\Omega\setminus B(a,d)}\frac{dx}{|x-a|^{2n-4}}\right),\end{aligned}$$

where we have used, for the first integral, the change of variables $y := l(x-a)$. For $n \geqslant 5$, we have

$$\int_{\Omega\setminus B(a,d)}\frac{dx}{|x-a|^{2n-4}} \leqslant c\int_d^\infty \frac{r^{n-1}}{r^{2n-4}}dr = c\int_d^\infty \frac{1}{r^{n-3}}dr \leqslant \frac{c}{d^{n-4}},$$

and

$$\begin{aligned}\frac{n-2}{2}c_0^2\int_{B(0,\lambda d)}\frac{1-|y|^2}{(1+|y|^2)^{n-1}}dy &= -\frac{n-2}{2}c_0^2\int_{\mathbb{R}^n}\frac{|y|^2-1}{(1+|y|^2)^{n-1}}dy\\ &\quad + O\left(\int_{\mathbb{R}^n\setminus B(0,\lambda d)}\frac{1}{(1+|y|^2)^{n-2}}dy\right)\\ &= -c_2 + O\left(\frac{1}{(\lambda d)^{n-4}}\right).\end{aligned}$$

Thus, for $n \geqslant 5$, we obtain

$$\int_\Omega P\delta_{a,\lambda}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda_1} = -\frac{c_2}{\lambda^2} + O\left(\frac{1}{(\lambda d)^{n-2}}\right).$$

However for $n = 4$, since Ω is bounded and $d_a \geqslant c > 0$, we derive that

$$\int_{\Omega\setminus B(a,d)}\frac{dx}{|x-a|^{2n-4}} \leqslant \int_{\Omega\setminus B(a,c)}\frac{dx}{|x-a|^4} = c,$$

and

$$\begin{aligned}\int_{B(0,\lambda d)}\frac{1-|y|^2}{(1+|y|^2)^{n-1}}dy &= \text{meas}(\mathbb{S}^3)\int_0^{\lambda d}\frac{1-r^2}{(1+r^2)^3}\\ &= \omega_3\left(\frac{1}{2}\int_1^{1+(\lambda d)^2}\frac{(2-t)(t-1)}{t^3}dt\right) \quad (t = 1+r^2).\\ &= \frac{1}{2}\omega_3\int_1^{1+(\lambda d)^2}\left(-\frac{1}{t}+\frac{3}{t^2}-\frac{2}{t^3}\right)dt\\ &= \frac{1}{2}\omega_3\left(-\ln\left(1+(\lambda d)^2\right)+O(1)\right)\\ &= -\frac{1}{2}\omega_3\left(2\ln\lambda + O(1)\right)\\ &= -\omega_3\ln\lambda + O(1).\end{aligned}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

This completes the proof . □

Theorem 4.7. *Let $n \geqslant 3, a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large. Then, for each $v \perp \partial P\delta_{a,\lambda}/\partial\lambda$, it holds:*

$$\left|\int_\Omega P\delta_{a,\lambda}^{4/n-2}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}v\right| \leqslant c\|v\| \begin{cases} (\lambda d)^{(2-n)} & \text{if } n \leqslant 5, \\ (\lambda d)^{-4}\ln^{2/3}(\lambda d) & \text{in } n = 6, \\ (\lambda d)^{-(n+2)/2} & \text{if } n \geqslant 7. \end{cases}$$

Proof. Observe that,

$$\int P\delta_{a,\lambda}^{\frac{4}{n-2}}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}v = \int_\Omega P\delta_{a,\lambda}^{\frac{4}{n-2}}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}v - \int_\Omega P\delta_{a,\lambda}^{\frac{4}{n-2}}\lambda\frac{\partial\theta_{a,\lambda}}{\partial\lambda}v.$$

Using Propositions 3.4, 3.3 and Theorem 4.2, we get

$$\begin{aligned}\int_\Omega P\delta_{a,\lambda}^{\frac{4}{n-2}}\left|\lambda\frac{\partial\theta_{a,\lambda}}{\partial\lambda}\right||v| &\leqslant c\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}|v| \\ &\leqslant c\|v\| \times \begin{cases} (\lambda d)^{(2-n)} & \text{if } n \leqslant 5, \\ (\lambda d)^{-4}\ln^{2/3}(\lambda d) & \text{if } n = 6, \\ (\lambda d)^{-(n+2)/2} & \text{if } n \geqslant 7. \end{cases}\end{aligned} \tag{45}$$

For the other integral, we get

$$\begin{aligned}\int_\Omega P\delta_{a,\lambda}^{\frac{4}{n-2}}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}v &= \int_\Omega (\delta_{a,\lambda}-\theta_{a,\lambda})^{\frac{4}{n-2}}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}v \\ &= \int \delta_{a,\lambda}^{\frac{4}{n-2}}\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}v + O\left(\int_\Omega \delta_{a,\lambda}^{\frac{6-n}{n-2}}\theta_{a,\lambda}\left|\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}\right||v|\right) \\ &= \frac{n-2}{n+2}\int_\Omega -\Delta\left(\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right)v + O\left(\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}|v|\right). \\ &= \frac{n-2}{n+2}\left\langle\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}, v\right\rangle + O\left(\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}|v|\right).\end{aligned}$$

The scalar product is zero and the integral is computed in (45). Thus the proof is complete. □

Theorem 4.8. *Let $n \geqslant 4$, $a \in \Omega$ and $\lambda > 0$ be such that λd_a is very large. For each $j \in \{1, \ldots, n\}$, it holds:*

(a) $\displaystyle\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j} = O\left(\frac{1}{(\lambda d_a)^{n+1}}\right)$,

(b) $\displaystyle\int_\Omega P\delta_{a,\lambda}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j} = 2\left\langle P\delta_{a,\lambda}, \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right\rangle + O\left(\frac{1}{(\lambda d_a)^n}\right)$,

(c) $\displaystyle\left\langle P\delta_{a,\lambda}, \frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j}\right\rangle = \frac{\tilde{c}}{2}\frac{\partial H(a_i a)}{\partial a} + O\left(\frac{1}{(\lambda d_a)^n}\right)$, *where $\tilde{c}$ is defined in Theorem 4.5.*

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Proof. (a) Using Theorem 3.2, observe that:

$$\begin{aligned}\int_{B(a,d_a)} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j} &= (n-2)c_0^{\frac{2n}{n-2}} \int_{B(a,d_a)} \frac{\lambda^{n+1}(x_j-a_j)}{\left(1+\lambda^2|x-a|^2\right)^{n+1}}dx.\\ &= (n-2)c_0^{\frac{2n}{n-2}} \int_{B(0,d_a)} \frac{\lambda^{n+1}y_j}{\left(1+\lambda^2|y|^2\right)^{n+1}}dy,\end{aligned}$$

by using the change of variable $y=(x-a)$. Note that the function in the last integral is even with respect to the variable y_j. Thus we get

$$\int_{B(a,d_a)} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j} = 0. \tag{46}$$

Furthermore, we have

$$\begin{aligned}\left|\int_{\Omega\setminus B(a,d_a)} \delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j}\right| &\leqslant c\int_{\Omega\setminus B(a,d_a)} \frac{\lambda^{n+1}|x-a|}{\lambda^{2n+2}|x-a|^{2n+2}}dx\\ &\leqslant \frac{c}{\lambda^{n+1}}\int_{\mathbb{R}^n\setminus B(a,d_a)} \frac{1}{|x-a|^{2n+1}}dx\\ &\leqslant \frac{c}{\lambda^{n+1}}\int_{d_a}^{\infty} \frac{1}{r^{n+2}}dr \leqslant \frac{c}{\lambda^{n+1}d_a^{n+1}}.\end{aligned} \tag{47}$$

Combining Eqs. (46) and (47).
(b) It follows that

$$\begin{aligned}\int_\Omega P\delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j} &= \int_\Omega (\delta_{a,\lambda}-\theta_{a,\lambda})^{\frac{n+2}{n-2}} \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\\ &= \int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}}\lambda\frac{\partial P\delta_{a,\lambda}}{\partial a_j} - \frac{n+2}{n-2}\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}\left(\frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j} - \frac{1}{\lambda}\frac{\partial \theta_{a,\lambda}}{\partial a_j}\right)\\ &+ O\left(\int_\Omega \delta_{a,\lambda}^{\frac{6-n}{n-2}}\theta_{a,\lambda}^2\left|\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right|\right).\end{aligned} \tag{48}$$

For the first integral in Eq. (48), using Theorem 2.3 and Eq. (10), we derive that

$$\begin{aligned}\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j} &= \int_\Omega (-\Delta P\delta_{a,\lambda})\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\\ &= \int_\Omega \nabla P\delta_{a,\lambda}\nabla\left(\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right)\\ &= \left\langle P\delta_{a,\lambda}, \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right\rangle.\end{aligned} \tag{49}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Concerning the second integral in Eq. (49), it will be divided into two pieces. For the first one, using Eq. (10) and Assertion (a), we get

$$\begin{aligned}\frac{n+2}{n-2}\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j}\theta_{a,\lambda} &= \frac{n+2}{n-2}\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j}\left(\theta_{a,\lambda}-\delta_{a,\lambda}\right)+O\left(\frac{1}{(\lambda d_a)^{n+1}}\right)\\ &= \int_{\Omega}\left(-\Delta\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right)\left(-P\delta_{a,\lambda}\right)+O\left(\frac{1}{(\lambda d_a)^{n+1}}\right)\\ &= -\int_{\Omega}\nabla\left(\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right)\nabla P\delta_{a,\lambda}+O\left(\frac{1}{(\lambda d_a)^{n+1}}\right)\\ &= -\left\langle P\delta_{a,\lambda},\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right\rangle+O\left(\frac{1}{(\lambda d_a)^{n+1}}\right).\end{aligned} \tag{50}$$

For the second piece, using Propositions 3.3 and 3.5, we get

$$\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}\frac{1}{\lambda}\left|\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right|\leqslant c\left|\frac{1}{\lambda}\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right|_{L^\infty(B)}|\theta_{a,\lambda}|_{L^\infty(B)}^{\frac{2}{n-2}}\int_B\delta_{a,\lambda}^{\frac{n}{n-2}}+c\int_{\Omega\setminus B}\delta_{a,\lambda}^{\frac{2n}{n-2}}.$$

where $B:=B(a,d_a/2)$. Now, observe that, for $x\in B\left(a,d_a/2\right)$, it follows that $d_a/2\leqslant d_x\leqslant(3/2)d_a$. Thus, using again Proposition 3.5, we obtain

$$\left|\frac{1}{\lambda}\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right|_{L^\infty(B)}\leq\frac{c}{\lambda^{n/2}d_a^{n-1}}.$$

Hence, using Assertion (b) of Theorem 4.4, Proposition 3.3 and Eqs.(16),

$$\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}\frac{1}{\lambda}\left|\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right|\leqslant\frac{c}{\lambda^{n/2}d_a^{n-1}}\frac{1}{(\lambda d_a^2)}\frac{\ln(\lambda d_a)}{\lambda^{n/2}}+\frac{c}{(\lambda d_a)^n}\leqslant\frac{c}{(\lambda d_a)^n}. \tag{51}$$

Thus the second integral in Eq. (48) becomes

$$\frac{n+2}{n-2}\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}\left(\frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a_j}-\frac{1}{\lambda}\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right)=-\left\langle P\delta_{a,\lambda},\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right\rangle+O\left(\frac{1}{(\lambda d_a)^n}\right). \tag{52}$$

Now, for the last integral in Eq. (48), using Proposition 3.5 and Eq. (36), we obtain

$$\int_{\Omega}\delta_{a,\lambda}^{\frac{6-n}{n-2}}\theta_{a,\lambda}^2\left|\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right|\leqslant\int_B\delta_{a,\lambda}^{\frac{6-n}{n-2}}\theta_{a,\lambda}^2\frac{1}{\lambda}\left|\frac{\partial\delta_{a,\lambda}}{\partial a_j}\right|+\int_B\delta_{a,\lambda}^{\frac{6-n}{n-2}}\theta_{a,\lambda}^2\frac{1}{\lambda}\left|\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right|+c\int_{\Omega\setminus B}\delta_{a,\lambda}^{\frac{2n}{n-2}} \tag{53}$$

The last integral in Eq. (53) is computed in Eq. (16). For the second one, it can be deduced from Eq. (51). Indeed, using Proposition 3.3, we have:

$$\int_{\Omega}\delta_{a,\lambda}^{\frac{6-n}{n-2}}\theta_{a,\lambda}^2\frac{1}{\lambda}\left|\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right|\leqslant\int_{\Omega}\delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}\frac{1}{\lambda}\left|\frac{\partial\theta_{a,\lambda}}{\partial a_j}\right|\leqslant\frac{c}{(\lambda d_a)^n}. \tag{54}$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Now, for the first integral in Eq. (53), using Theorem 3.2 and Proposition 3.3, we have

$$\begin{aligned}
\int_B \delta_{a,\lambda}^{\frac{6-n}{n-2}} \theta_{a,\lambda}^2 \frac{1}{\lambda}\left|\frac{\partial \delta_{a,\lambda}}{\partial a_j}\right| &\leqslant c \int_B \delta_{a,\lambda}^{\frac{6-n}{n-2}} \theta_{a,\lambda}^2 \frac{1}{\lambda\,|x-a|}\delta_{a,\lambda} \\
&\leqslant c \int_B \frac{1}{\lambda\,|x-a|}\theta_{a,\lambda}^2 \delta_{a,\lambda}^{\frac{4}{n-2}} \\
&\leqslant c\,|\theta_{a,\lambda}|_{L^\infty(B)}^{n/n-2} \int_B \frac{1}{\lambda\,|x-a|}\delta_{a,\lambda}^{n/n-2} \\
&\leqslant \frac{c}{(\lambda d_a^2)^{n/2}} \int_{B(a,d_a/2)} \frac{1}{\lambda\,|x-a|} \frac{\lambda^{n/2}}{\left(1+\lambda^2|x-a|^2\right)^{n/2}} \\
&\leqslant \frac{c}{(\lambda d_a^2)^{n/2}} \frac{1}{\lambda^{n/2}} \int_0^{\lambda d_a/2} \frac{r^{n-2}}{(1+r^2)^{n/2}} dr \\
&\leqslant \frac{c}{(\lambda d_a)^n}.
\end{aligned} \tag{55}$$

Combining Eqs. (55), (54) and (16), the Eq. (53) becomes

$$\int_\Omega \delta_{a,\lambda}^{\frac{6-n}{n-2}} \theta_{a,\lambda}^2 \frac{1}{\lambda}\left|\frac{\partial \delta_{a,\lambda}}{\partial a_j}\right| \leq \frac{c}{(\lambda d_a)^n}. \tag{56}$$

Hence, the equations (48), (49), (50), (51) and (56) imply that

$$\int_\Omega P\delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j} = 2\left\langle P\delta_{a,\lambda}, \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right\rangle + O\left(\frac{1}{(\lambda d_a)^n}\right).$$

(c) Using Proposition 3.5, observe that

$$\begin{aligned}
\left\langle P\delta_{a,\lambda}, \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right\rangle &= \int_\Omega \nabla P\delta_{a,\lambda}\cdot\nabla\left(\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right) \\
&= \int_\Omega -\Delta P\delta_{a,\lambda}\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j} \\
&= \int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}}\left(\frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j} - \frac{c_0}{\lambda^{n/2}}\frac{\partial H(a,\cdot)}{\partial a_j} + O\left(\frac{1}{\lambda^{\frac{n+4}{2}} d_a^{n+1}}\right)\right).
\end{aligned} \tag{57}$$

The integral in Eq. (57) will be divided in three pieces. The first one is computed in Assertion (a) and the second one can be deduced from Assertion (c) of Theorem 4.4 and we have:

$$\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}} \frac{1}{\lambda^{\frac{n+4}{2}} d_a^{n+1}} \leqslant \frac{c}{\lambda^{(n-2)/2}} \frac{1}{\lambda^{(n+4)/2} d_a^{n+1}} = \frac{c}{(\lambda d_a)^{n+1}}. \tag{58}$$

For the last piece, expanding $\frac{\partial H(a,\cdot)}{\partial a_j}$ around a in $B := B(a, d_a/2)$ and using Eq. (??) and Theorem 2.4, we get

$$\frac{\partial H}{\partial a_j}(a,y) = \frac{\partial H}{\partial a_j}(a,a) + O\left(\frac{|y-a|}{d_a^n}\right).$$

*Corresponding Author: Ayidh Munahi Bati Alqahtani

Thus, we obtain

$$\frac{c_0}{\lambda^{n/2}}\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}}(y)\frac{\partial H}{\partial a_j}(a,y)dy = \frac{c_0}{\lambda^{n/2}}\frac{\partial H}{\partial a_j}(a,a)\int_B \delta_{a,\lambda}^{\frac{n+2}{n-2}} + O\left(\int_{\Omega\setminus B}\delta_{a,\lambda}^{\frac{n+2}{n-2}}\frac{1}{\lambda^{n/2}d_a^{n-1}}\right)$$
$$+ O\left(\frac{1}{\lambda^{n/2}d_a^n}\int_B |y-a|\delta_{a,\lambda}^{\frac{n+2}{n-2}}\right)$$
$$= \frac{\tilde{c}}{\lambda^{n-1}}\frac{\partial H}{\partial a_j}(a,a) + O\left(\frac{1}{(\lambda d_a)^{n+1}} + \frac{1}{\lambda^{n/2}d_a^n}\int_B |y-a|\delta_{a,\lambda}^{\frac{n+2}{n-2}}\right),$$

where we have Theorem 4.4. In addition, we have

$$\int_{\mathbb{R}^n}|y-a|\delta_{a,\lambda}^{\frac{n+2}{n-2}}(y)dy \leqslant c\int_{\mathbb{R}^n}\frac{\lambda^{\frac{n+2}{2}}|y-a|}{(1+\lambda^2|y-a|^2)^{(n+2)/2}}dy$$
$$\leqslant \frac{c}{\lambda^{n/2}}\int_{\mathbb{R}^n}\frac{|x|}{(1+|x|^2)^{(n+2)/2}}dx$$
$$\leqslant \frac{c}{\lambda^{n/2}},$$

by using the change of variables $x = \lambda(y-a)$. Hence we obtain

$$\frac{c_0}{\lambda^{n/2}}\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}}(y)\frac{\partial H}{\partial a_j}(a,y)dy = \frac{\tilde{c}}{\lambda^{n-1}}\frac{\partial H}{\partial a_j}(a,a) + O\left(\frac{1}{(\lambda d_a)^n}\right). \tag{59}$$

Combining (58), (59), (57) and Assertion (a), we derive that

$$\left\langle P\delta_{a,\lambda}, \frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a_j}\right\rangle = -\frac{\tilde{c}}{\lambda^{n-1}}\frac{\partial H}{\partial a_j}(a,a) + O\left(\frac{1}{(\lambda d_a)^n}\right)$$

which completes the proof. □

5. CONCLUSIONS

As we know that the Brezis-Nirenberg problem is introduced first fundamental results about the existence of positive solutions were obtained by H. Brezis and L. Nirenberg in 1983. Several authors explained that there exists a positive solution of (P_ε) for every $\varepsilon \in (0,\lambda_1(\Omega)),\lambda_1(\Omega)$ being the first eigenvalue of −4 in Ω with Dirichlet boundary conditions. In this paper, we introduced and investigated some asymptotic analysis for a family of sign-changing solutions of this problem. We studied the approximate solutions for the point-wise estimate of the derivative of the approximate solution and the estimating some integrals involving the function $P\delta_{a,\lambda}$ and its derivatives with respect to λ and the point a.

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