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Evaluation of Delay Term and Noise Term for Approximate Solution of Stochastic Delay Differential Equation without Interpolation Techniques

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Abstract:
The main objective of this study is to tackle the setbacks encountered by many researchers in the evaluations of the delay term and noise term of Stochastic Delay Differential Equation (SDDE) in order to obtain its numerical solution. These setbacks affect the preservation of the accuracy and efficiency of the numerical solution of Stochastic Delay Differential Equation (SDDE) which need to be addressed. Three new mathematical expressions are developed for the evaluations of the delay term and noise term different from the interpolation techniques been used by other researchers which causes setbacks in obtaining and preserving the accuracy of the numerical solutions. These mathematical expressions developed for the evaluations of the delay term and the noise term together with the discrete schemes of Hybrid Extended Block Backward Differentiation Formulae Method (HEBBDFM) are incorporated into some examples of Stochastic Delay Differential Equation (SDDE) for its numerical solutions. The discrete schemes of the method (HEBBDFM) are obtained by continuous formulations of multistep collocation method by matrix inversion approach. The basic properties for the convergence and stability analysis of the method were analyzed and proved satisfactory. The analysis of the numerical results with its comparison and graphical presentations proved that the method performs better by producing the Least Minimum Absolute Random Errors (LMAREs) at Lesser Computational Processing Unit Time (LCPUT) than other existing methods in terms of accuracy and efficiency using the developed mathematical expressions. Thus, this study recommends the three new mathematical expressions developed for the evaluations of the delay term and noise term different from the interpolation techniques suitable for obtaining better approximate solutions of SDDE.

Keywords: *Stochastic delay differential equation, Hybrid Extended Block Backward Differentiation Formulae Method, Delay term, Noise term, Absolute Random Error*

Mathematics Subject Classification 2020: 60H10; 91G60; 62-08; 60H40; 60H25.

1. Introduction

Stochastic delay differential equation (SDDE) is a stochastic differential equation where the increment of the process depends not only on current state but also on the history or future part which contains the random values of the system being modeled as a noise term. A noise term is a stochastic process on any family of random variables $\{X_t, t \epsilon T\}$ where X_t is, in practice, the observation at time t, and T is the time range involved which its applications can be seen in applied sciences, economics, and engineering. From the statistical point of view, a stochastic process is any probability process, that is, any process 'running along' in time and controlled by probabilistic laws. Mathematically, a stochastic process may be defined as a collection of random variables which are ordered in time and defines at a set of time points which may be continuous or discrete [1].

Several researchers such as [2, 3, 4] used Euler-Maruyama scheme to formulate continuous split-step schemes of SDDE on a continuous interval $t_0 \leq t \leq t_a$ for the numerical solutions and encountered some setbacks in the use of interpolation techniques in evaluating the delay term and noise term. [5] adopted Malliavin calculus and a refined Lindeberg principle in solving some examples of SDDE and encountered setbacks in obtaining week approximations of

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uniform error bounds. [6] applied residual power series method (RPSM) to achieve approximate solution with high degree of accuracy for a system of multi-pantograph type delay differential equations and encountered challenges in obtaining accurate approximate solutions of the modeled system. The setbacks as studied by [7, 8, 9] revealed that; the order of the interpolating polynomials should be at least the same with numerical methods which is very difficult to carry out to obtain numerical solution of any modeled system. There are emergence of discontinuities when the interpolation techniques switches from initial function to past values during the numerical integration and the discontinuities observed when the initial function is not fully compatible with the rest of the modeled system. In order to overcome these setbacks, three new mathematical expressions are developed for the evaluations of the delay term and noise term.

[10] developed a general equation for stochastic delay differential equations (SDDEs) using Adomian Decomposition Method (ADM) as analytic method for analytic solutions. The governing equation for SDDE takes into account the current state and the history part of the system being modeled and was expressed as;

$$dy(t) = f(y(t), y(t-\tau), t)dt + g(y(t), y(t-\tau), t)d\xi(t), \quad for\ t > 0, \tau > 0. \quad y(t) = \lambda(t),\ for\ t > 0. \qquad (1)$$

where $\lambda(t)$ is the initial function, $y(t)$ is the stochastic process of the current state, t is the time, τ is called the delay, $(t-\tau)$ is the delay term, and $y(t-\tau)$ is the solution of the delay term on the drift part of (1). $\xi(t)$ is the Standard Brownian Motion with its differential equivalence as $d\xi(t)$. The drift part of the equation (1) $dy(t) = f(y(t), y(t-\tau), t)dt$ is deterministic and takes care of the average time rate of (1). The volatility or diffusion part $dy(t) = \beta(y(t), y(t-\tau), t)d\xi(t)$ is stochastic, which takes care of the random change in the modeled system in (1). The rest of the paper is organized as follows: Section 2 presents derivation of the discrete schemes of the method for step numbers and analysis of important properties of the method. Evaluation of delay term, noise term, and numerical computations are done in Section 3. The results are presented and discussed in Section 4, while Section 5 concludes the work.

2. Derivation of the Method and Analysis of Basic Properties of the Method

By the *k*-step multistep collocation method and matrix inversion techniques developed by [11] the discrete schemes of the Hybrid Extended Block Backward Differentiation Formulae Method (HEBBDFM) for step numbers k = 2, 3, and 4 are derived and presented as;

For $k = 2$ of (HEBBDFM)

$$y_{n+1} = \frac{248}{45}hf_{n+1} - \frac{1517}{10}hf_{n+2} - \frac{4333}{90}hf_{n+3} - \frac{21194}{45}hf_{n+\frac{5}{2}} + \frac{19088}{45}hf_{n+\frac{9}{4}} + \frac{3632}{15}hf_{n+\frac{11}{4}} + y_n$$

$$y_{n+2} = -\frac{73}{1736}y_n + \frac{1809}{1736}y_{n+1} + \frac{1859461}{156240}hf_{n+2} - \frac{288754}{9765}hf_{n+\frac{9}{4}} + \frac{404263}{13020}hf_{n+\frac{5}{2}} - \frac{21502}{1395}hf_{n+\frac{11}{4}} + \frac{66763}{22320}hf_{n+3}$$

$$y_{n+\frac{9}{4}} = -\frac{10675}{253952}y_n + \frac{264627}{253952}y_{n+1} + \frac{6090795}{507904}hf_{n+2} - \frac{465975}{15872}hf_{n+\frac{9}{4}} + \frac{3932955}{126976}hf_{n+\frac{5}{2}} - \frac{244215}{15872}hf_{n+\frac{11}{4}} + \frac{1516995}{507904}hf_{n+3}$$

$$y_{n+\frac{5}{2}} = -\frac{4671}{111104}y_n + \frac{115775}{111104}y_{n+1} + \frac{2662935}{222208}hf_{n+2} - \frac{202995}{6944}hf_{n+\frac{9}{4}} + \frac{1729095}{55552}hf_{n+\frac{5}{2}} - \frac{15285}{992}hf_{n+\frac{11}{4}} + \frac{94905}{31744}hf_{n+3}$$

$$y_{n+\frac{11}{4}} = -\frac{10675}{253952}y_n + \frac{264627}{253952}y_{n+1} + \frac{274022287}{22855680}hf_{n+2} - \frac{20901419}{714240}hf_{n+\frac{9}{4}} + \frac{59597461}{1904640}hf_{n+\frac{5}{2}} - \frac{10922219}{714240}hf_{n+\frac{11}{4}} + \frac{68201287}{22855680}hf_{n+3}$$

$$y_{n+3} = -\frac{73}{1736}y_n + \frac{1809}{1736}y_{n+1} + \frac{207957}{17360}hf_{n+2} - \frac{31698}{1085}hf_{n+\frac{9}{4}} + \frac{135333}{4340}hf_{n+\frac{5}{2}} - \frac{2334}{155}hf_{n+\frac{11}{4}} + \frac{7611}{2480}hf_{n+3} \qquad (2)$$

For $k = 3$ of (HEBBDFM)

$$y_{n+1} = -\frac{9373517}{11408760}hf_{n+1} - \frac{696013}{207432}hf_{n+3} + \frac{292489}{103716}hf_{n+4} + \frac{428528}{43215}hf_{n+\frac{7}{2}} - \frac{2804480}{285219}hf_{n+\frac{15}{4}} - \frac{3151}{23048}y_n + \frac{26199}{23048}y_{n+2}$$

$$y_{n+2} = \frac{9373517}{4371885}hf_{n+2} - \frac{4075204}{624555}hf_{n+3} + \frac{927397}{208185}hf_{n+4} + \frac{10557376}{624555}hf_{n+\frac{7}{2}} - \frac{70227968}{4371885}hf_{n+\frac{15}{4}} - \frac{1193}{13879}y_n + \frac{15072}{13879}y_{n+1}$$

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$$y_{n+3}=\frac{144377}{9373517}y_n-\frac{1564731}{9373517}y_{n+1}+\frac{10793871}{9373517}y_{n+2}+\frac{23388822}{9373517}hf_{n+3}-\frac{45109440}{9373517}hf_{n+\frac{7}{2}}+\frac{40722432}{9373517}hf_{n+\frac{15}{4}}-\frac{10904274}{9373517}hf_{n+4}$$

$$y_{n+\frac{7}{2}}=\frac{9134775}{599905088}y_n-\frac{24820803}{149976272}y_{n+1}+\frac{690053525}{599905088}y_{n+2}+\frac{402702825}{149976272}hf_{n+3}-\frac{40561395}{9373517}hf_{n+\frac{7}{2}}+\frac{38661000}{9373517}hf_{n+\frac{15}{4}}-\frac{335171025}{299952544}hf_{n+4}$$

$$y_{n+\frac{15}{4}}=\frac{146309933}{9598481408}y_n-\frac{24839325}{149976272}y_{n+1}+\frac{11041888275}{9598481408}y_{n+2}+\frac{12864892425}{4799240704}hf_{n+3}-\frac{5038612425}{1199810176}hf_{n+\frac{7}{2}}+\frac{319538835}{74988136}hf_{n+\frac{15}{4}}-\frac{676350675}{599905088}hf_{n+4}$$

$$y_{n+4}=\frac{142641}{9373517}y_n-\frac{1550656}{9373517}y_{n+1}+\frac{10781532}{9373517}y_{n+2}+\frac{25178016}{9373517}hf_{n+3}-\frac{39817728}{9373517}hf_{n+\frac{7}{2}}+\frac{41828352}{9373517}hf_{n+\frac{15}{4}}-\frac{9706980}{9373517}hf_{n+4} \qquad (3)$$

For $k=4$ of (HEBBDFM)

$$y_{n+1}=-\frac{17939}{22302}hf_{n+1}+\frac{8284}{11151}hf_{n+4}+\frac{1325}{7434}hf_{n+5}-\frac{128}{189}hf_{n+\frac{9}{2}}-\frac{4594}{33453}y_n+\frac{2294}{1239}y_{n+2}-\frac{3413}{4779}y_{n+3}$$

$$y_{n+2}=-\frac{71756}{32355}hf_{n+2}-\frac{2638}{2157}hf_{n+4}-\frac{1706}{6471}hf_{n+5}+\frac{33856}{32355}hf_{n+\frac{9}{2}}+\frac{1565}{19413}y_n-\frac{683}{719}y_{n+1}+\frac{36289}{19413}y_{n+3}$$

$$y_{n+3}=\frac{215268}{173263}hf_{n+3}-\frac{1253502}{1212841}hf_{n+4}-\frac{219294}{1212841}hf_{n+5}+\frac{939840}{1212841}hf_{n+\frac{9}{2}}+\frac{34291}{1212841}y_n-\frac{43929}{173263}y_{n+1}+\frac{1486053}{1212841}y_{n+2}$$

$$y_{n+4}=-\frac{1657}{125573}y_n+\frac{1982}{17939}y_{n+1}-\frac{55890}{125573}y_{n+2}+\frac{24178}{17939}y_{n+3}+\frac{152304}{125573}hf_{n+4}-\frac{75648}{125573}hf_{n+\frac{9}{2}}+\frac{15804}{125573}hf_{n+5}$$

$$y_{n+\frac{9}{2}}=-\frac{230125}{18369536}y_n+\frac{1937925}{18369536}y_{n+1}-\frac{7890939}{18369536}y_{n+2}+\frac{24552675}{18369536}y_{n+3}+\frac{13480425}{9184768}hf_{n+4}-\frac{94815}{287024}hf_{n+\frac{9}{2}}+\frac{978075}{9184768}hf_{n+5}$$

$$y_{n+5}=-\frac{1702}{125573}y_n+\frac{2025}{17939}y_{n+1}-\frac{56650}{125573}y_{n+2}+\frac{24250}{17939}y_{n+3}+\frac{171900}{125573}hf_{n+4}+\frac{9600}{125573}hf_{n+\frac{9}{2}}+\frac{36240}{125573}hf_{n+5} \qquad (4)$$

2.2. Analysis of Basic Properties of the Method

The order, error constant, consistency, zero stability, and region of absolute stability of (2), (3), and (4) are analyzed using the conditions proposed by [12] and [13].

2.2.1. Order and Error Constant

[12] studied and proved that the Linear Multistep Method is said to be of order e if $c_0=c_1=0,\dots,c_p=0$ but $c_{p+1}\neq 0$ and c_{p+1} is the error constant.

The order and error constants for (2) are obtained as follows:

$$c_0=c_1=c_2=c_3=c_4=c_5=c_6=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T \text{ but}$$

$$\left(\frac{67031}{1935360},-\frac{6280327}{3359784960},-\frac{6800355}{3640655872},-\frac{2976175}{1592786944},-\frac{26228807}{14042529792},-\frac{232669}{124436480}\right)^T.$$

Therefore, (2) has order $p=6$ and error constant,

$$\frac{67031}{1935360},-\frac{6280327}{3359784960},-\frac{6800355}{3640655872},-\frac{2976175}{1592786944},-\frac{26228807}{14042529792},-\frac{232669}{124436480}.$$

Following the same approach to (3), we obtained:

$$c_0=c_1=c_2=c_3=c_4=c_5=c_6=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T \text{ but}$$

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$\left(-\frac{1835179}{139394304},-\frac{6496457}{419700960},\frac{17421171}{5249169520},\frac{250360775}{76787851264},\frac{16053110535}{4914422480896},\frac{610393}{187470340}\right)^T$. Therefore, Eq. (3) has order $p=6$ and error constant,

$$-\frac{1835179}{139394304},-\frac{6496457}{419700960},\frac{17421171}{5249169520},\frac{250360775}{76787851264},\frac{16053110535}{4914422480896},\frac{610393}{187470340}.$$

Applying the same approach to (4), we obtained:

$c_0=c_1=c_2=c_3=c_4=c_5=c_6=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$ but

$$\left(-\frac{180601}{9366840},-\frac{48523}{2174256},\frac{7688973}{679190960},-\frac{216271}{35160440},-\frac{1684125}{293912576},-\frac{45505}{7032088}\right)^T.$$

Therefore, (4) has order $p=6$ and error constant,

$$-\frac{180601}{9366840},-\frac{48523}{2174256},\frac{7688973}{679190960},-\frac{216271}{35160440},-\frac{1684125}{293912576},-\frac{45505}{7032088}.$$

2.2.2 Consistency

According to [12], a Linear Multistep Method is said to be consistent if the order p is greater than 1 i.e. $p \geq 1$. Since the order of our proposed method HEBBDFM as analyzed using the discrete schemes (2), (3), and (4) is greater than 1 i.e. $p \geq 1$, therefore method is consistent.

2.2.3 Zero Stability Analysis

In [13], a Linear Multistep Method is said to be zero stable if no roots $r_s, s=1,2,3,\ldots,n$ of the first characteristic polynomial $P(r)$ expressed as $P(r)=\det(rT_2^{(i)}-T_1^{(i)})$ is greater than 1 which satisfies $|r_i| \leq 1$ and the roots $|r_i|$ is simple or distinct.

The zero stability for (2) is examined as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1809}{1736} & 1 & 0 & 0 & 0 & 0 \\ -\frac{264627}{253952} & 0 & 1 & 0 & 0 & 0 \\ -\frac{115775}{111104} & 0 & 0 & 1 & 0 & 0 \\ -\frac{264627}{253952} & 0 & 0 & 0 & 1 & 0 \\ -\frac{1809}{1736} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{9}{4}} \\ y_{n+\frac{5}{2}} \\ y_{n+\frac{11}{4}} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{73}{1736} \\ 0 & 0 & 0 & 0 & 0 & \frac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \frac{4671}{111104} \\ 0 & 0 & 0 & 0 & 0 & \frac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \frac{73}{1736} \end{pmatrix} \begin{pmatrix} y_{n-\frac{11}{4}} \\ y_{n-\frac{5}{2}} \\ y_{n-\frac{9}{4}} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$+h\begin{pmatrix} \frac{248}{45} & -\frac{1517}{10} & \frac{19088}{45} & -\frac{21194}{15} & \frac{3632}{15} & -\frac{4333}{90} \\ 0 & \frac{1859461}{156240} & -\frac{288754}{9765} & \frac{404263}{13020} & -\frac{21502}{1395} & \frac{66763}{22320} \\ 0 & \frac{6090795}{507904} & -\frac{465975}{15872} & \frac{3932955}{126976} & -\frac{244215}{15872} & \frac{1516995}{507904} \\ 0 & \frac{2662935}{222208} & -\frac{202995}{6944} & \frac{1729095}{55552} & -\frac{15285}{992} & \frac{94905}{31744} \\ 0 & \frac{274022287}{22855680} & -\frac{20901419}{714240} & \frac{59597461}{1904640} & -\frac{10922219}{714240} & \frac{68201287}{22855680} \\ 0 & \frac{207957}{17360} & -\frac{31698}{1085} & \frac{135333}{4340} & -\frac{2334}{155} & \frac{7611}{2480} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{9}{4}} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{11}{4}} \\ f_{n+3} \end{pmatrix}$$

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$$+h\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} f_{n-\frac{11}{4}}\\ f_{n-\frac{5}{2}}\\ f_{n-\frac{9}{4}}\\ f_{n-2}\\ f_{n-1}\\ f_{n} \end{pmatrix}$$

where

$$T_2^{(1)}=\begin{pmatrix} 1&0&0&0&0&0\\ -\frac{1809}{1736}&1&0&0&0&0\\ -\frac{264627}{253952}&0&1&0&0&0\\ -\frac{115775}{111104}&0&0&1&0&0\\ -\frac{264627}{253952}&0&0&0&1&0\\ -\frac{1809}{1736}&0&0&0&0&1 \end{pmatrix},\ T_1^{(1)}=\begin{pmatrix} 0&0&0&0&0&-1\\ 0&0&0&0&0&\frac{73}{1736}\\ 0&0&0&0&0&\frac{10675}{253952}\\ 0&0&0&0&0&\frac{4671}{111104}\\ 0&0&0&0&0&\frac{10675}{253952}\\ 0&0&0&0&0&\frac{73}{1736} \end{pmatrix}$$

and

$$U_2^{(1)}=\begin{pmatrix} \frac{248}{45}&-\frac{1517}{10}&\frac{19088}{45}&-\frac{21194}{15}&\frac{3632}{15}&-\frac{4333}{90}\\ 0&\frac{1859461}{156240}&-\frac{288754}{9765}&\frac{404263}{13020}&-\frac{21502}{1395}&\frac{66763}{22320}\\ 0&\frac{6090795}{507904}&-\frac{465975}{15872}&\frac{3932955}{126976}&-\frac{244215}{15872}&\frac{1516995}{507904}\\ 0&\frac{2662935}{222208}&-\frac{202995}{6944}&\frac{1729095}{55552}&-\frac{15285}{992}&\frac{94905}{31744}\\ 0&\frac{274022287}{22855680}&-\frac{20901419}{714240}&\frac{59597461}{1904640}&-\frac{10922219}{714240}&\frac{68201287}{22855680}\\ 0&\frac{207957}{17360}&-\frac{31698}{1085}&\frac{135333}{4340}&-\frac{2334}{155}&\frac{7611}{2480} \end{pmatrix}$$

$$\begin{aligned} P(r)&=\det\left(rT_2^{(1)}-T_1^{(1)}\right)\\ &=\left|rT_2^{(1)}-T_1^{(1)}\right|=0. \end{aligned} \qquad (5)$$

Now we have,

$$P(r)=\left| r\begin{pmatrix} 1&0&0&0&0&0\\ -\frac{1809}{1736}&1&0&0&0&0\\ -\frac{264627}{253952}&0&1&0&0&0\\ -\frac{115775}{111104}&0&0&1&0&0\\ -\frac{264627}{253952}&0&0&0&1&0\\ -\frac{1809}{1736}&0&0&0&0&1 \end{pmatrix}-\begin{pmatrix} 0&0&0&0&0&-1\\ 0&0&0&0&0&\frac{73}{1736}\\ 0&0&0&0&0&\frac{10675}{253952}\\ 0&0&0&0&0&\frac{4671}{111104}\\ 0&0&0&0&0&\frac{10675}{253952}\\ 0&0&0&0&0&\frac{73}{1736} \end{pmatrix}\right|$$

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$$= \left| \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 \\ -\frac{1809}{1736}r & r & 0 & 0 & 0 & 0 \\ -\frac{264627}{253952}r & 0 & r & 0 & 0 & 0 \\ -\frac{115775}{111104}r & 0 & 0 & r & 0 & 0 \\ -\frac{264627}{253952}r & 0 & 0 & 0 & r & 0 \\ -\frac{1809}{1736}r & 0 & 0 & 0 & 0 & r \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{73}{1736} \\ 0 & 0 & 0 & 0 & 0 & \frac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \frac{4671}{111104} \\ 0 & 0 & 0 & 0 & 0 & \frac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \frac{73}{1736} \end{pmatrix} \right|$$

$$\Rightarrow P(r) = \begin{vmatrix} r & 0 & 0 & 0 & 0 & 1 \\ -\frac{1809}{1736}r & r & 0 & 0 & 0 & -\frac{73}{1736} \\ -\frac{264627}{253952}r & 0 & r & 0 & 0 & -\frac{10675}{253952} \\ -\frac{115775}{111104}r & 0 & 0 & r & 0 & -\frac{4671}{111104} \\ -\frac{264627}{253952}r & 0 & 0 & 0 & r & -\frac{10675}{253952} \\ -\frac{1809}{1736}r & 0 & 0 & 0 & 0 & r-\frac{73}{1736} \end{vmatrix}.$$

Using Maple (18) software, we obtain:

$P(r) = r^5(r+1),$

$\Rightarrow r^5(r+1) = 0,$

$\Rightarrow r_1 = -1, r_2 = 0, r_3 = 0, r_4 = 0, r_5 = 0, r_6 = 0$ .Since $|r_i| < 1,\ i = 1,2,3,4,5,6$ (2) is zero stable.

Implementing the same approach, then **(3)** is presented as:

$$\begin{pmatrix} 1 & -\frac{26199}{23048} & 0 & 0 & 0 & 0 \\ -\frac{15072}{13879} & 1 & 0 & 0 & 0 & 0 \\ \frac{1564731}{9373517} & -\frac{10793871}{9373517} & 1 & 0 & 0 & 0 \\ \frac{24820803}{149976272} & -\frac{690053525}{599905088} & 0 & 1 & 0 & 0 \\ \frac{24839325}{149976272} & -\frac{11041888275}{9598481408} & 0 & 0 & 1 & 0 \\ \frac{1550656}{9373517} & -\frac{10781532}{9373517} & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+\frac{15}{4}} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{3151}{23048} \\ 0 & 0 & 0 & 0 & 0 & \frac{1193}{13879} \\ 0 & 0 & 0 & 0 & 0 & -\frac{144377}{9373517} \\ 0 & 0 & 0 & 0 & 0 & -\frac{9134775}{599905088} \\ 0 & 0 & 0 & 0 & 0 & -\frac{146309933}{9598481408} \\ 0 & 0 & 0 & 0 & 0 & -\frac{142641}{9373517} \end{pmatrix} \begin{pmatrix} y_{n-\frac{15}{4}} \\ y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

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$$+h\begin{pmatrix} -\frac{9373517}{11408760} & 0 & -\frac{696013}{207432} & \frac{428528}{43215} & -\frac{2804480}{285219} & \frac{292489}{103716} \\ 0 & \frac{9373517}{4371885} & -\frac{4075204}{624555} & \frac{10557376}{624555} & -\frac{70227968}{4371885} & \frac{927397}{208185} \\ 0 & 0 & \frac{23388822}{9373517} & -\frac{45109440}{9373517} & \frac{40722432}{9373517} & -\frac{10904274}{9373517} \\ 0 & 0 & \frac{402702825}{149976272} & -\frac{40561395}{9373517} & \frac{38661000}{9373517} & -\frac{335171025}{299952544} \\ 0 & 0 & \frac{12864892425}{4799240704} & -\frac{5038612425}{1199810176} & \frac{319538835}{74988136} & -\frac{676350675}{599905088} \\ 0 & 0 & \frac{25178016}{9373517} & -\frac{39817728}{9373517} & \frac{41828352}{9373517} & -\frac{9706980}{9373517} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+\frac{15}{4}} \\ f_{n+4} \end{pmatrix}$$

$$+h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-\frac{11}{4}} \\ f_{n-\frac{7}{2}} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$

where $T_2^{(2)} = \begin{pmatrix} 1 & -\frac{26199}{23048} & 0 & 0 & 0 & 0 \\ -\frac{15072}{13879} & 1 & 0 & 0 & 0 & 0 \\ \frac{1564731}{9373517} & -\frac{10793871}{9373517} & 1 & 0 & 0 & 0 \\ \frac{24820803}{149976272} & -\frac{690053525}{599905088} & 0 & 1 & 0 & 0 \\ \frac{24839325}{149976272} & -\frac{11041888275}{9598481408} & 0 & 0 & 1 & 0 \\ \frac{1550656}{9373517} & -\frac{10781532}{9373517} & 0 & 0 & 0 & 1 \end{pmatrix}, T_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{3151}{23048} \\ 0 & 0 & 0 & 0 & 0 & \frac{1193}{13879} \\ 0 & 0 & 0 & 0 & 0 & -\frac{144377}{9373517} \\ 0 & 0 & 0 & 0 & 0 & -\frac{9134775}{599905088} \\ 0 & 0 & 0 & 0 & 0 & -\frac{146309933}{9598481408} \\ 0 & 0 & 0 & 0 & 0 & -\frac{142641}{9373517} \end{pmatrix}$

and $U_2^{(2)} = \begin{pmatrix} -\frac{9373517}{11408760} & 0 & -\frac{696013}{207432} & \frac{428528}{43215} & -\frac{2804480}{285219} & \frac{292489}{103716} \\ 0 & \frac{9373517}{4371885} & -\frac{4075204}{624555} & \frac{10557376}{624555} & -\frac{70227968}{4371885} & \frac{927397}{208185} \\ 0 & 0 & \frac{23388822}{9373517} & -\frac{45109440}{9373517} & \frac{40722432}{9373517} & -\frac{10904274}{9373517} \\ 0 & 0 & \frac{402702825}{149976272} & -\frac{40561395}{9373517} & \frac{38661000}{9373517} & -\frac{335171025}{299952544} \\ 0 & 0 & \frac{12864892425}{4799240704} & -\frac{5038612425}{1199810176} & \frac{319538835}{74988136} & -\frac{676350675}{599905088} \\ 0 & 0 & \frac{25178016}{9373517} & -\frac{39817728}{9373517} & \frac{41828352}{9373517} & -\frac{9706980}{9373517} \end{pmatrix}$

$$P(r) = \det\left(rT_2^{(2)} - T_1^{(2)}\right) = \left|rT_2^{(2)} - T_1^{(2)}\right| = 0. \qquad (6)$$

Now we have,

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$$P(r)=\left|r\begin{pmatrix} 1 & -\frac{26199}{23048} & 0 & 0 & 0 & 0 \\ -\frac{15072}{13879} & 1 & 0 & 0 & 0 & 0 \\ \frac{1564731}{9373517} & -\frac{10793871}{9373517} & 1 & 0 & 0 & 0 \\ \frac{24820803}{149976272} & -\frac{690053525}{599905088} & 0 & 1 & 0 & 0 \\ \frac{24839325}{149976272} & -\frac{11041888275}{9598481408} & 0 & 0 & 1 & 0 \\ \frac{1550656}{9373517} & -\frac{10781532}{9373517} & 0 & 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{3151}{23048} \\ 0 & 0 & 0 & 0 & 0 & \frac{1193}{13879} \\ 0 & 0 & 0 & 0 & 0 & -\frac{144377}{9373517} \\ 0 & 0 & 0 & 0 & 0 & -\frac{9134775}{599905088} \\ 0 & 0 & 0 & 0 & 0 & -\frac{146309933}{9598481408} \\ 0 & 0 & 0 & 0 & 0 & -\frac{142641}{9373517} \end{pmatrix}\right|$$

$$=\left|\begin{pmatrix} r & -\frac{26199}{23048}r & 0 & 0 & 0 & 0 \\ -\frac{15072}{13879}r & r & 0 & 0 & 0 & 0 \\ \frac{1564731}{9373517}r & -\frac{10793871}{9373517}r & r & 0 & 0 & 0 \\ \frac{24820803}{149976272}r & -\frac{690053525}{599905088}r & 0 & r & 0 & 0 \\ \frac{24839325}{149976272}r & -\frac{11041888275}{9598481408}r & 0 & 0 & r & 0 \\ \frac{1550656}{9373517}r & -\frac{10781532}{9373517}r & 0 & 0 & 0 & r \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{3151}{23048} \\ 0 & 0 & 0 & 0 & 0 & \frac{1193}{13879} \\ 0 & 0 & 0 & 0 & 0 & -\frac{144377}{9373517} \\ 0 & 0 & 0 & 0 & 0 & -\frac{9134775}{599905088} \\ 0 & 0 & 0 & 0 & 0 & -\frac{146309933}{9598481408} \\ 0 & 0 & 0 & 0 & 0 & -\frac{142641}{9373517} \end{pmatrix}\right|$$

$$\Rightarrow P(r)=\begin{vmatrix} r & -\frac{26199}{23048}r & 0 & 0 & 0 & -\frac{3151}{23048} \\ -\frac{15072}{13879}r & r & 0 & 0 & 0 & -\frac{1193}{13879} \\ \frac{1564731}{9373517}r & -\frac{10793871}{9373517}r & r & 0 & 0 & \frac{144377}{9373517} \\ \frac{24820803}{149976272}r & -\frac{690053525}{599905088}r & 0 & r & 0 & \frac{9134775}{599905088} \\ \frac{24839325}{149976272}r & -\frac{11041888275}{9598481408}r & 0 & 0 & r & \frac{146309933}{9598481408} \\ \frac{1550656}{9373517}r & -\frac{10781532}{9373517}r & 0 & 0 & 0 & r+\frac{142641}{9373517} \end{vmatrix}$$

Using Maple (18) software, we obtain:

$$P(r)=-\frac{9373517}{39985399}r^5(r+1),$$

$$\Rightarrow -\frac{9373517}{39985399}r^5(r+1)=0,$$

$\Rightarrow r_1=-1, r_2=0, r_3=0, r_4=0, r_5=0, r_6=0$ .Since $|r_i|<1,\ i=1,2,3,4,5,6$ (3) is zero stable.

With the same procedure (4) can be presented as follows:

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$$\begin{pmatrix} 1 & -\frac{2294}{1239} & \frac{3413}{4779} & 0 & 0 & 0 \\ \frac{683}{719} & 1 & -\frac{36289}{19413} & 0 & 0 & 0 \\ \frac{43929}{173263} & -\frac{1486053}{1212841} & 1 & 0 & 0 & 0 \\ -\frac{1982}{17939} & \frac{55890}{125573} & -\frac{24178}{17939} & 1 & 0 & 0 \\ -\frac{1937925}{18369536} & \frac{7890939}{18369536} & -\frac{24552675}{18369536} & 0 & 1 & 0 \\ -\frac{2025}{17939} & \frac{56650}{125573} & -\frac{24250}{17939} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{4594}{33453} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1565}{19413} \\ 0 & 0 & 0 & 0 & 0 & -\frac{34291}{1212841} \\ 0 & 0 & 0 & 0 & 0 & \frac{1657}{125573} \\ 0 & 0 & 0 & 0 & 0 & \frac{230125}{18369536} \\ 0 & 0 & 0 & 0 & 0 & \frac{1702}{125573} \end{pmatrix} \begin{pmatrix} y_{n-\frac{9}{2}} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$+h \begin{pmatrix} -\frac{17939}{22302} & 0 & 0 & \frac{8284}{11151} & -\frac{128}{189} & \frac{1325}{7434} \\ 0 & -\frac{71756}{32355} & 0 & -\frac{2638}{2157} & \frac{33856}{32355} & -\frac{1706}{6471} \\ 0 & 0 & \frac{215268}{173263} & -\frac{1253502}{1212841} & -\frac{939840}{1212841} & -\frac{219294}{1212841} \\ 0 & 0 & 0 & \frac{152304}{125573} & -\frac{75648}{125573} & \frac{15804}{125573} \\ 0 & 0 & 0 & \frac{13480425}{9184768} & -\frac{94815}{287024} & \frac{978075}{9184768} \\ 0 & 0 & 0 & \frac{171900}{125573} & \frac{9600}{125573} & \frac{36240}{125573} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+\frac{9}{2}} \\ f_{n+5} \end{pmatrix}$$

$$+h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{9}{2}} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

where

$$T_2^{(3)} = \begin{pmatrix} 1 & -\frac{2294}{1239} & \frac{3413}{4779} & 0 & 0 & 0 \\ \frac{683}{719} & 1 & -\frac{36289}{19413} & 0 & 0 & 0 \\ \frac{43929}{173263} & -\frac{1486053}{1212841} & 1 & 0 & 0 & 0 \\ -\frac{1982}{17939} & \frac{55890}{125573} & -\frac{24178}{17939} & 1 & 0 & 0 \\ -\frac{1937925}{18369536} & \frac{7890939}{18369536} & -\frac{24552675}{18369536} & 0 & 1 & 0 \\ -\frac{2025}{17939} & \frac{56650}{125573} & -\frac{24250}{17939} & 0 & 0 & 1 \end{pmatrix}, T_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{4594}{33453} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1565}{19413} \\ 0 & 0 & 0 & 0 & 0 & -\frac{34291}{1212841} \\ 0 & 0 & 0 & 0 & 0 & \frac{1657}{125573} \\ 0 & 0 & 0 & 0 & 0 & \frac{230125}{18369536} \\ 0 & 0 & 0 & 0 & 0 & \frac{1702}{125573} \end{pmatrix}$$

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and

$$U_2^{(3)}=\begin{pmatrix} -\frac{17939}{22302} & 0 & 0 & \frac{8284}{11151} & -\frac{128}{189} & \frac{1325}{7434} \\ 0 & -\frac{71756}{32355} & 0 & -\frac{2638}{2157} & \frac{33856}{32355} & -\frac{1706}{6471} \\ 0 & 0 & \frac{215268}{173263} & -\frac{1253502}{1212841} & -\frac{939840}{1212841} & -\frac{219294}{1212841} \\ 0 & 0 & 0 & \frac{152304}{125573} & -\frac{75648}{125573} & \frac{15804}{125573} \\ 0 & 0 & 0 & \frac{13480425}{9184768} & -\frac{94815}{287024} & \frac{978075}{9184768} \\ 0 & 0 & 0 & \frac{171900}{125573} & \frac{9600}{125573} & \frac{36240}{125573} \end{pmatrix}$$

$$P(r)=\det\left(rT_2^{(3)}-T_1^{(3)}\right) \qquad (7)$$

$$=\left|rT_2^{(3)}-T_1^{(3)}\right|=0.$$

Now we have,

$$P(r)=\left| r\begin{pmatrix} 1 & -\frac{2294}{1239} & \frac{3413}{4779} & 0 & 0 & 0 \\ \frac{683}{719} & 1 & -\frac{36289}{19413} & 0 & 0 & 0 \\ \frac{43929}{173263} & -\frac{1486053}{1212841} & 1 & 0 & 0 & 0 \\ -\frac{1982}{17939} & \frac{55890}{125573} & -\frac{24178}{17939} & 1 & 0 & 0 \\ -\frac{1937925}{18369536} & \frac{7890939}{18369536} & -\frac{24552675}{18369536} & 0 & 1 & 0 \\ -\frac{2025}{17939} & \frac{56650}{125573} & -\frac{24250}{17939} & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{4594}{33453} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1565}{19413} \\ 0 & 0 & 0 & 0 & 0 & -\frac{34291}{1212841} \\ 0 & 0 & 0 & 0 & 0 & \frac{1657}{125573} \\ 0 & 0 & 0 & 0 & 0 & \frac{230125}{18369536} \\ 0 & 0 & 0 & 0 & 0 & \frac{1702}{125573} \end{pmatrix} \right|$$

$$=\left| \begin{pmatrix} r & -\frac{2294}{1239}r & \frac{3413}{4779}r & 0 & 0 & 0 \\ \frac{683}{719}r & r & -\frac{36289}{19413}r & 0 & 0 & 0 \\ \frac{43929}{173263}r & -\frac{1486053}{1212841}r & r & 0 & 0 & 0 \\ -\frac{1982}{17939}r & \frac{55890}{125573}r & -\frac{24178}{17939}r & r & 0 & 0 \\ -\frac{1937925}{18369536}r & \frac{7890939}{18369536}r & -\frac{24552675}{18369536}r & 0 & r & 0 \\ -\frac{2025}{17939}r & \frac{56650}{125573}r & -\frac{24250}{17939}r & 0 & 0 & r \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{4594}{33453} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1565}{19413} \\ 0 & 0 & 0 & 0 & 0 & -\frac{34291}{1212841} \\ 0 & 0 & 0 & 0 & 0 & \frac{1657}{125573} \\ 0 & 0 & 0 & 0 & 0 & \frac{230125}{18369536} \\ 0 & 0 & 0 & 0 & 0 & \frac{1702}{125573} \end{pmatrix} \right|$$

$$\Rightarrow P(r)=\begin{pmatrix} r & -\frac{2294}{1239}r & \frac{3413}{4779}r & 0 & 0 & -\frac{4594}{33453} \\ \frac{683}{719}r & r & -\frac{36289}{19413}r & 0 & 0 & \frac{1565}{19413} \\ \frac{43929}{173263}r & -\frac{1486053}{1212841}r & r & 0 & 0 & \frac{34291}{1212841} \\ -\frac{1982}{17939}r & \frac{55890}{125573}r & -\frac{24178}{17939}r & r & 0 & -\frac{1657}{125573} \\ -\frac{1937925}{18369536}r & \frac{7890939}{18369536}r & -\frac{24552675}{18369536}r & 0 & r & -\frac{230125}{18369536} \\ -\frac{2025}{17939}r & \frac{56650}{125573}r & -\frac{24250}{17939}r & 0 & 0 & r-\frac{1702}{125573} \end{pmatrix}$$

Using Maple (18) software, we obtain:

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$$P(r)=\frac{51489235360}{154349784183}r^5(r+1),$$

$$\Rightarrow \frac{51489235360}{154349784183}r^5(r+1)=0,$$

$\Rightarrow r_1=-1, r_2=0, r_3=0, r_4=0, r_5=0, r_6=0$ .Since $|r_i|<1,\ i=1,2,3,4,5,6$ (4) is zero stable.

2.2.4 Convergence

Adopting the theorem established by [13], the necessary and sufficient condition for a linear multistep method to be convergent is that it must be consistent and zero-stable. Since the discrete schemes (2), (3), and (4) are both consistent and zero stable, therefore the method is convergent.

2.2.5 Region of Absolute Stability

The P- and Q- regions of absolute stability of the numerical methods for discrete schemes (2), (3), and (4) are plotted using Maple 18 and
MATLAB software and are presented in figure 1 to 4 below;

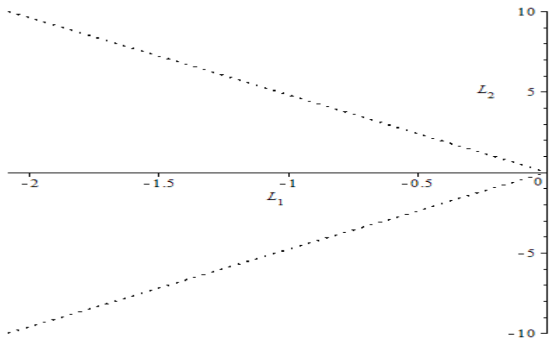

Fig.1: Region of P -stability (HEBBDFM) in (2)

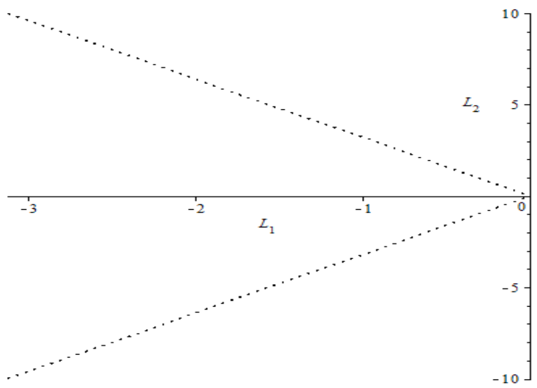

Fig.2: Region of P -stability (HEBBDFM) in (3)

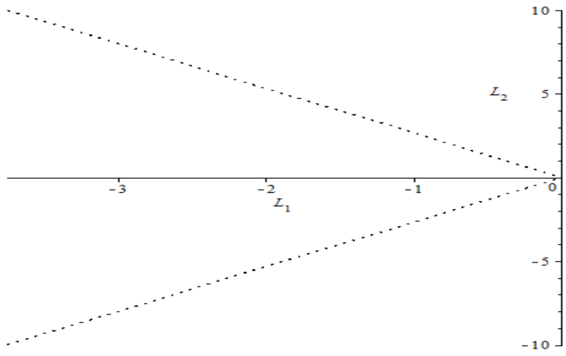

Fig.3: Region of P -stability (HEBBDFM) in (4)

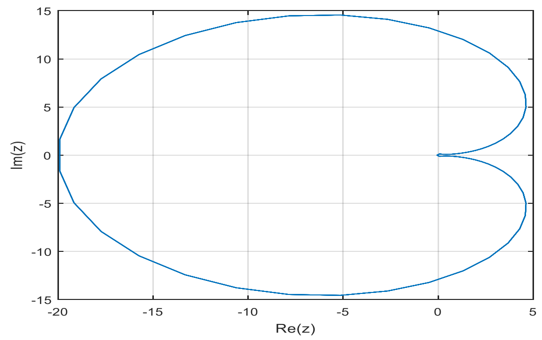

Fig.4: Region of Q -stability (HEBBDFM) in (2)

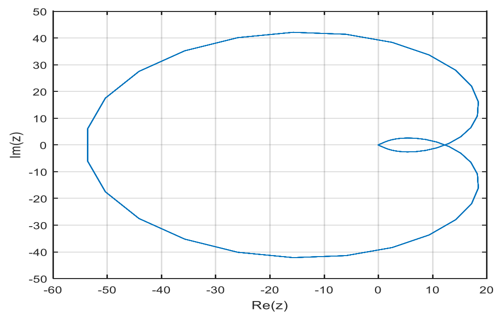

Fig.5: Region of Q -stability (HEBBDFM) in (3)

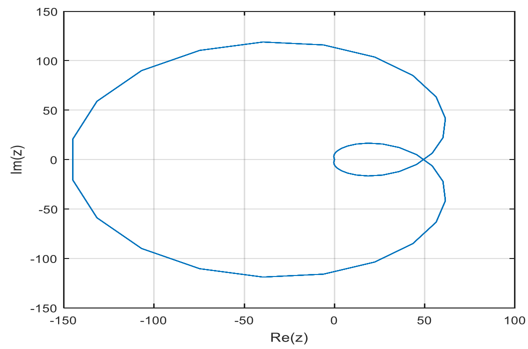

Fig.6: Region of Q -stability (HEBBDFM) in (4)

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The P-stability regions in Figs 1 to 3 lie inside the open-ended region while the Q-stability regions in Figs. 4 to 6 lie inside the enclosed region. Therefore, the region of absolute stability of our proposed method is satisfied. Satisfactorily, the analyses of the proposed method have satisfied the necessary and sufficient conditions for a numerical method to be successfully applied for numerical solutions of any modeled system as developed and stated by [12] and [13].

3. Formulations and Evaluations of the Delay Term $(t-\tau)$ and the Noise Term $d\psi(t)$

The delay term $(t+\tau)$ and the noise terms $d\psi(t)$ are derived and evaluated using newly developed mathematical expressions different from the interpolation techniques used by other researchers as cited in the literature for better and faster evaluations, computations, performances, and accurate results. The newly developed mathematical expressions are developed and prove to be mutually independent random variables having normal distribution $N(0,1)$ for their numerical evaluations and implementations. The newly developed mathematical expressions for the evaluations of the delay term and the noise term are incorporated into some examples of the Stochastic Delay Differential Equation (SDDE) with the derived discrete schemes of (2), (3) and (4) before its numerical experiments at constant step size $h = 0.01$ to obtain the numerical solutions of $dy(t)$ with the help of Maple 18 software.

Researchers such as [14, 15, 16] applied the formula developed by [17] for the evaluation of the delay term of first-order delay differential equations and discovered that it gives lesser accurate results, it takes more time to compute, and cannot be adequately use in solving different classes of DDEs such as Stochastic delay differential equations, Advanced Stochastic Delay Differential Equations, Riccati delay differential equations, Partial delay differential equations, and Stochastic partial delay differential equations. It is highly required that an accurate mathematical expressions need to be develop to address these challenges and that of the setbacks encountered by the researchers in the use of interpolation techniques for evaluations of the delay term and the noise term.

In the sequel, one states;

Theorem 3.1

Let the current state and history part of the drift and stochastic coefficients of (1) be represented as α and β, then the corresponding values of the functions $y(t-\tau)$ and $y(t-\tau)d\psi(t)$ with an accurate formula for the evaluation of the delay term $(t-\tau)$ is given as:

$$\xi_{n+u}(t) = \frac{n}{c}\left((cq + (n+u+m-1)h)\right), c \neq 0. \quad (8)$$

Where $u \in (-k, k), k$ is a step number $m = \frac{\tau}{h} \in \mathbb{Z}, \tau = mh, \tau$ is the delay term, $n = 0,1,2,\ldots, N-1$and N is the number of solutions in the giving interval which is implemented to approximate the delay term $(t-\tau)$ at the point $t = t_n - \tau$ using the values of ξ_{n+u} at $t_n - \tau > 0$ whenever $t_n - \tau > 0$ where $\xi_{n+u}(t)$ is the approximation to $y(t_n - \tau)$.

Proof

The expression in (8) is formulated using the idea of the sum of arithmetic progression (AP) as developed by [18] of the form:

$$\xi_n = \frac{n}{c}(cq + (n-1)h). \quad (9)$$

Incorporating the delay term in (9), one obtains equation (8) as required.

The results of the above expression in (8) are obtained using Maple 18 with $n = 0,1,2,3,\ldots, N-1$ which shall be incorporated into the Stochastic Delay Differential Equation (SDDE) before its evaluation at constant step size h to obtain the numerical solutions of $dy(t)$.

Theorem 3.2

Let $\psi(t)$ be a normalized Brownian Motion Process for hyperbolic equivalence of Euler's exponential function with the mean μ and the volatility σ given as $N(0,1)$. Then the discrete noise term $d\psi(t)$ is given as:

$$d\psi(t) = \frac{1}{\sqrt{2\pi}}\left(e^{\frac{-3t^2}{2}} - te^{\frac{-t^2-2}{2}}\right), \qquad for\ 0 \leq t \leq 30. \quad (10)$$

Proof

Using the idea of normalized Brownian Motion Process for hyperbolic function, the continuous time stochastic process is given as,

$$\psi(t) = \frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}sinht, \qquad for\ 0 \leq t \leq 30. \quad (11)$$

Then by differentiating and discretizing (11) using hyperbolic equivalence of Euler's exponential function, one arrived at (10) as required.

Theorem 3.3

Let theorem 3.2 exists, then the modified discrete noise term, $d\psi(t)$, using Iterative Adomian Decomposition Method (IADM) is given as

$$d\psi(t) = \frac{V_0}{\sqrt{2\pi}} + \sum_{h=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{V_h(e^t + e^{-t})}{2}, \qquad for\ 0 \leq t \leq 30. \quad (12)$$

Proof

[19] obtained the continuous noise term $d\psi(t)$ as a derivative equivalence of normalized induced Brownian Motion $\psi(t)$ developed by [20] for the analytical approximate solution of the linear Stochastic Differential Equations (SDE) using Iterative Adomian Decomposition Method (IADM). The normalized induced Brownian motion and the continuous noise term developed by these two scholars are presented as:

$$\psi(t) = \frac{V_0}{\sqrt{2\pi}}t + \sum_{h=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{V_h}{h}sin(ht), \qquad for\ t \in [0,\pi]. \quad (13)$$

Differentiating and discretizing (13) using hyperbolic equivalence of Euler's exponential function one arrived at (12) as required.

where $V_h = 0,1,2,3,\ldots, V_{h+1}, \forall, h = 1,2,3,\ldots,30$ are randomly generated values within the time interval $0 \leq t \leq 30$. If one

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further assumes $V_0 = 0$, by Riemann-Stieltjes theorem in [21], then (10) and (12) become;

$$\begin{cases} A_h(t) = \sqrt{\frac{2}{\pi h}} \int_0^t V_h sinht\, dw. \\ B_e(t) = \sqrt{\frac{2}{\pi}} \int_0^t V_h cosht\, dw. \end{cases} \quad (14)$$

Lemma 1: Let the density of a Brownian bridge expressed by [22] exist:

$f(x) = \frac{1}{t(1-t)\sqrt{2\pi}} e^{-\frac{x^2}{2t(1-t)^2}}$, where $\mu = 0$ and $\sigma = t(1-t)$.

(15)

Then, the function $F(t) := \int_0^1 \frac{1}{\sqrt{t(1-t)}} exp\left\{-\frac{(x-t)^2}{2t(1-t)}\right\} dt$

(16)

Is constant for $x \in [0,1]$ given as

$$F(t) = \int_{-\infty}^{\infty} (t^2+1)^{-1} e^{-\frac{t^2}{2}} He_0(t)dt, \quad for\ x \in [0,1].$$

where He_0 is the Hermite polynomial developed in [23].

Remark: Equation (15) is equivalent to the density function derived in [18].

Sketch of Proof

Put $t = cos^2 m$. Then$F(t) := 2\int_0^{\frac{\pi}{2}} exp\left\{-\frac{(cos^2 m - t)^2}{2sin^2 m cos^2 m}\right\} dm = \int_0^{\pi} exp\left\{-\frac{(cosm-2t+1)^2}{2sin^2 m}\right\} dm,$

Hence

$F(t) = F(1-t), x\epsilon[0,1]\ and\ F(^1/_2) = F(1) = F(0).$

(17)

Following the proposition 1 developed by [24], the proof is completed.

Theorem 3.4: Let lemma 1 and equation (14) exist, such that

$A_0 = \frac{1}{\sqrt{2\pi}} F(t), A_h(t) = \sqrt{\frac{2}{\pi}} \int_0^t V_h cos\, ht\, dw = B_h(t) = \sqrt{\frac{2}{\pi h}} \int_0^t V_h sin\, ht\, dw, h = 1,2, \ldots$ (18)

Then $A_0, A_h(2\pi), B_e(2\pi), h\epsilon N \cup \{0\}, e \in N$ are mutually independent random variables having normal distribution $N(0,1)$.

Proof: Following the proof of asymptotics of solutions to semi-linear stochastic wave equations by [25], Let $m, n \in R; h, e \in N$ be arbitrary. For mutually independent events, we have $exp[t(mA_h(t) + nB_e(t))] =$

$1 + nm \int_0^t exp[t(mB_e(t) + nA_h(t))]\, dB_e(t) +$
$nm \int_0^t exp[t(eB_h(t) + nA_h(t))]\, dA_h(t) -$
$\frac{1}{2} mn \int_0^t exp[t(mB_e(t) + eA_h(t))]\, d[B_e, A_h] -$
$\frac{1}{2} m^2 \int_0^t exp[t(mB_e(t) + nA_h(t))]\, d[B_e] -$
$\frac{1}{2} n^2 \int_0^t exp[t(mB_e(t) + nA_h(t))]\, d[A_h], \forall t \in [0, \infty).$ (19)

We observe that the second and third term on the right-hand side are continuous martingales; thus, taking expectations on both sides and denoting $exp[t(mB_e(t) + nA_h(t))]$ by $f(t)$, we get

$E[f(t)] = 1 - \frac{1}{2\pi} mnE\left[\int_0^t f(t) sinhtcoshtdt\right] - \frac{1}{2\pi} m^2 E\left[\int_0^t f(t) sin^2 htdt\right] - \frac{1}{2\pi} n^2 E[\int_0^t f(t) cos^2 htdt].$

(20)

By Fubini's Theorem [26], we now have

$E[f(t)] = 1 - \frac{1}{2\pi} mn \int_0^t E[f(t)] \sin ht \cos ht\, dt - \frac{1}{2\pi} m^2 \int_0^t E[f(t)] sin^2 htdt - \frac{1}{2\pi} n^2 \int_0^t E[f(t)] cos^2 htdt.$ (21)

So we have that $E[f(t)]$ is the solution of the differential equation of the form:

$\frac{dE[f(t)]}{dt} = -\frac{1}{2\pi} E[f(t)](mn\, sinht\, cosht + m^2 sin^2 ht + n^2 cos^2 ht), E[f(0)] = 1.$ (22)

Thus we have:

$$E[f(t)] = \exp\left[-\frac{1}{2\pi}\int_0^t (mn \sin ht \cos ht + m^2 sin^2 ht + n^2\, cos\, ht)dt\right],$$

which implies

$$E[f(2\pi)] = \exp\left[-\frac{1}{2\pi}\int_0^{2\pi} (m^2 sin^2 ht + n^2 \cos ht)dt\right],$$

$= \exp[-\frac{1}{2\pi} sinm^2 + n^2 \int_0^{2\pi} sinht + \cos ht\, dt].$ (23)

From the above, we immediately have:

$E[exp[i(mB_e + nA_h)]] = E[exp[imB_e]]E[exp[inA_h]].$

(24)

Evaluating the integrals in (23) and through the existence of lemma 1, we arrived at (18) as required.

3.2. Numerical Implementation and Computations

In this section, the three newly developed mathematical expressions (8), (10), and (12) for the evaluations delay term and the noise term and the discrete schemes (2), (3), and (4) of the proposed method shall be incorporated into some stochastic delay differential equation (SDDE) before its numerical evaluation at constant step size h = 0.01 using Maple 18 software to obtain the approximate solutions of $dy(t)$ i.e. we present some numerical examples below:

Numerical Examples

Example 1

$dy(t) = (y(t+1+e^{-t}) + sin(t+1+e^{-t}) + cos(t))dt + (y(t+1+e^{-t}) + sin(t+1+e^{-t}) + cos(t))d\psi(t),\ 0 \le t \le 30.$

$y(t) = \sin(t), t \ge 0.$

Exact Solution $y(t) = \sin(t)$.

Example 2

$dy(t) = 1000(y(t) + 997e^{-3}y(t+1) + (1000 + 997e^{-3}))dt + (y(t) + 997e^{-3}y(t+1) + (1000 + 997e^{-3}))d\psi(t), 0 \le t \le 30.$

$y(t) = 1 + e^{-3t}, t \ge 0.$

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Exact Solution $y(t) = 1 + e^{-3t}$.

4. Results, Graphical Presentations and Discussions

The above examples are solved using the three newly developed mathematical expressions (8), (10) and (12) and the discrete schemes (2), (3), and (4) of the proposed method and the results of the absolute random errors obtained are presented in tables 4.1 to 4.4:

Table 4.1: Absolute Random Errors of Example 1 with the incorporation of Theorem 3.1 and Theorem 3.2 using the HEBBDFM for Step Numbers *k* = 2, 3 & 4.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error	K = 4 Absolute Random Error
1	0.151749006	0.251675413	0.351636052
2	0.133888662	0.391767148	0.434024768
3	0.533396698	0.564941926	0.719570845
4	0.120144171	0.337315884	0.610317974
5	0.250886544	0.454521134	0.779301502
6	0.423979366	0.642457089	0.62404228
7	0.055527009	0.208739098	0.604589222
8	0.184721682	0.373526988	0.450729529
9	0.00380954	0.205657906	0.156242056
10	0.043437365	0.164785168	0.308561015
11	0.39155517	0.587401949	0.765056243
12	0.10476663	0.148012845	0.195535686
13	0.092283747	0.302912025	0.453143165
14	0.172964715	0.371599274	0.67095625
15	0.117522643	0.432409239	0.534400138
16	0.043607255	0.29377271	0.477919129
17	0.222861176	0.504324613	0.722006875
18	0.122581634	0.32976314	0.511084818
19	0.083013112	0.417634184	0.478704103
20	0.153887366	0.453106725	0.791043882
21	0.124495372	0.175726126	0.423774518
22	0.002436898	0.162238412	0.278444508
23	0.173779485	0.400597475	0.599136746
24	0.237180686	0.518284464	0.775245405
25	0.087394605	0.312942756	0.513063203
26	0.220323043	0.438307904	0.619901938
27	0.229566692	0.313086126	0.513615757
28	0.045850163	0.149690804	0.270065984
29	0.048272082	0.244489553	0.448041535
30	0.273751706	0.568404579	0.973586293

CPU time of HEBBDFM for k = 2 is 0.01s, k = 3 is 0.03 and k = 3 is 0.05s

Table 4.2: Absolute Random Errors of Example 2 with the incorporation of Theorem 3.1 and Theorem 3.2 using the HEBBDFM for Step Numbers *k* = 2, 3 & 4.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error	K = 4 Absolute Random Error
1	0.106544924	0.206627655	0.306780996
2	0.00074618	0.20080951	0.30103032
3	0.17789138	0.277741069	0.353400231
4	0.095032886	0.32534714	0.369104522
5	0.155859468	0.214067633	0.35768996
6	0.133791545	0.196678392	0.330863395
7	0.113179304	0.275157699	0.374941095
8	0.286247054	0.342437322	0.542205252
9	0.201032457	0.335396398	0.428959298
10	0.278848147	0.329226807	0.475983214
11	0.370509132	0.409756453	0.568613855
12	0.353478693	0.407516171	0.552633919
13	0.386606521	0.486255307	0.585239984
14	0.477218367	0.576789992	0.676056757
15	0.476103234	0.577610428	0.639726042
16	0.369697382	0.42729852	0.535098541
17	0.318719959	0.423825727	0.518117715
18	0.316303671	0.424646437	0.516065894
19	0.216421883	0.109355657	0.029145497
20	0.214889697	0.10838427	0.038343675
21	0.206006142	0.100872082	0.110058612
22	0.305623844	0.20525367	0.105599035

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23	0.205683629	0.105045731	0.104826696
24	0.405452757	0.505135389	0.60474258
25	0.404545617	0.504541348	0.60453017
26	0.204508725	0.40450718	0.503507565
27	0.204516994	0.40452405	0.504456155
28	0.104495413	0.204446568	0.300445078
29	0.304440854	0.404442244	0.504439829
30	0.504438708	0.704444583	0.804438703

CPU time of HEBBDFM for k = 2 is 0.02s, k = 3 is 0.04 and k = 3 is 0.06s

Table 4.3: Absolute Random Errors of Example 1 with the incorporation of Theorem 3.1 and Theorem 3.3 using the HEBBDFM for Step Numbers *k* = 2, 3 & 4.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error	K = 4 Absolute Random Error
1	0.439930012	0.259930029	0.049929998
2	0.496486394	0.426486416	0.226486359
3	0.281692244	0.219261223	0.122514227
4	0.382255939	0.508783261	0.60261085
5	0.672946053	0.451958864	0.343561831
6	0.484741604	0.298238193	0.272864773
7	0.263778078	0.105857287	0.147684385
8	0.658281924	0.366985815	0.863837132
9	0.293493026	0.199266938	0.011446375
10	0.283153988	0.098687877	0.00939656
11	0.739727778	0.498886673	0.288579883
12	0.279385236	0.110901793	0.003527773
13	0.369020681	0.107807365	0.075262803
14	0.652486745	0.399373803	0.588929344
15	0.337673559	0.24001928	0.337607705
16	0.331227795	0.251321827	0.054223191
17	0.587337159	0.395451224	0.284788277
18	0.498905709	0.23622772	0.145760946
19	0.689322235	0.467155218	0.381026247
20	0.658203399	0.502452525	0.595281818
21	0.550911204	0.389441326	0.228966328
22	0.747482163	0.452575214	0.345426861
23	0.73703121	0.411223281	0.300238046
24	0.725265901	0.228719937	0.157179895
25	0.676026392	0.55257247	0.376853291
26	0.194158982	0.170294886	0.121319527
27	0.731047407	0.396188517	0.634893496
28	0.343303667	0.254681625	0.046617552
29	0.426482072	0.294229613	0.220938939
30	0.652917073	0.339711148	0.276373698

CPU time of HEBBDFM for k = 2 is 0.07s, k = 3 is 0.09 and k = 3 is 0.12s

Table 4.4: Absolute Random Errors of Example 2 with the incorporation of Theorem 3.1 and Theorem 3.3 using the HEBBDFM for Step Numbers *k* = 2, 3 & 4.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error	K = 4 Absolute Random Error
1	0.906544924	0.706627655	0.606780996
2	0.699253819	0.599190489	0.498969679
3	0.52210862	0.422258931	0.346599769
4	0.774967114	0.47465286	0.367897909
5	0.644140532	0.485932367	0.34231004
6	0.566208455	0.503321608	0.369136605
7	0.786820696	0.524842301	0.425058905
8	0.713752946	0.553318009	0.457794748
9	0.798967543	0.464603602	0.871040702
10	0.821151853	0.670773193	0.724016786
11	0.829490868	0.490243547	0.431386145
12	0.746521307	0.492483829	0.247366081
13	0.703393479	0.513744693	0.434760016
14	0.702781633	0.513210008	0.423943243
15	0.823896766	0.622389572	0.560273958
16	0.830302618	0.67270148	0.564901459
17	0.781280041	0.476174273	0.381882285
18	0.783696329	0.475353563	0.383934106

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19	0.783578117	0.390644343	0.290854503
20	0.885110303	0.39161573	0.291656325
21	0.893993858	0.391277883	0.279941388
22	0.794376156	0.39474633	0.194400965
23	0.794316371	0.294954269	0.195173304
24	0.794547243	0.294864611	0.19525742
25	0.795454383	0.295458652	0.19546983
26	0.695491275	0.19549282	0.095492435
27	0.695483006	0.19547595	0.195543845
28	0.795504587	0.395553432	0.295549207
29	0.795559146	0.395557756	0.295560171
30	0.895561292	0.495555417	0.395561297

CPU time of HEBBDFM for k = 2 is 0.06s, k = 3 is 0.10s and k = 3 is 0.14s

4.1Graphical Presentation of Absolute Errors

Using R and R – studio softwares, the graphs of Absolute Random Error Results of HEBBDFM for Examples1 and 2 above in tables 4.1 to 4.4 are plotted and presented as;

4.1.1 Graphical Presentations of the Absolute Random Errors for Implicit HEBBDFM after the Incorporations of the Theorems 3.1 and 3.2 for Examples 1 and 2

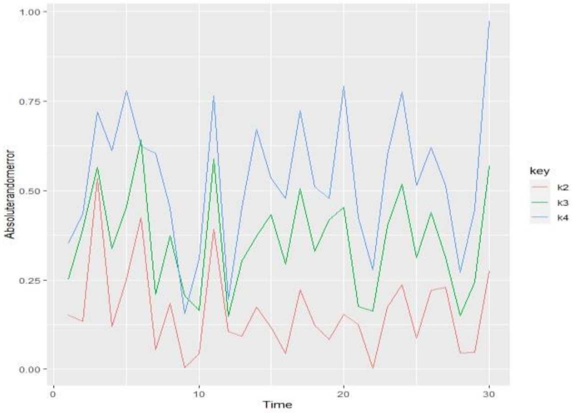

Fig.7: Absolute Random Error Results for Example 1 using HEBBDFM (as seen in the colors) against Time of future delay in days. The colorful lines represent the behavior or performance of the method for step numbers $k = 2, 3$ *and* 4 with different Absolute Random Errors.

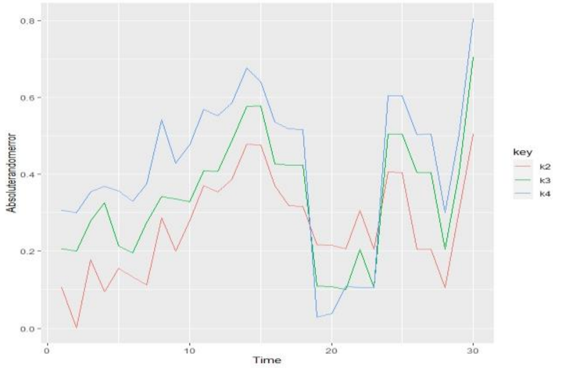

Fig.8: Absolute Random Error Results for Example 2 using HEBBDFM (as seen in the colors) against Time of future delay in days. The colorful lines represent the behavior or performance of the method for step numbers $k = 2, 3$ *and* 4 with different Absolute Random Errors.

4.1.2 Graphical Presentations of the Absolute Random Errors for Implicit HEBBDFM after the Incorporations of the Theorems 3.1 and 3.3 for Examples 1 and 2

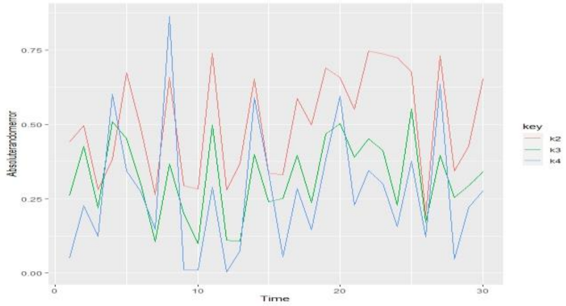

Fig.9: Absolute Random Error Results for Example 1 using HEBBDFM (as seen in the colors) against Time of future delay in days. The colorful lines represent the behavior or performance of the method for step numbers $k = 2, 3$ *and* 4 with different Absolute Random Errors.

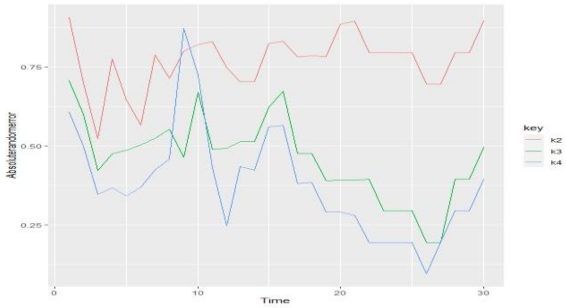

Fig.10: Absolute Random Error Results for Example 2 using HEBBDFM (as seen in the colors) against Time of future delay in days. The colorful lines represent the behavior or performance of the method for step numbers $k = 2, 3$ *and* 4 with different Absolute Random Errors.

4.2Comparison of Results

In order to determine the accuracy, efficiency, and advantage of our method HEBBDFM, we compared the Minimum Absolute Random Errors (MAREs) of our method with other existing methods in [9, 10, 18] below:

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TABLE 4.2.1: Comparison between the Minimum Absolute Random Errors (MAREs) of HEBBDFM after the incorporations of theorem 3.1 with theorems 3.2 and 3.3 for k = 2, 3 and 4 with [9, 10, 18] at constant step size h = 0.01 for Example 1.

Numerical Method	**Compared MAREs with** [9, 10, 18]
HEBBDFM MARE for k = 2 by theorems 3.1 and 3.2	2.44E-03
HEBBDFM MARE for k = 3 by theorems 3.1 and 3.2	1.48E-01
HEBBDFM MARE for k = 4 by theorems 3.1 and 3.2	1.56E-01
HEBBDFM MARE for k = 2 by theorems 3.1 and 3.3	1.94E-01
HEBBDFM MARE for k = 3 by theorems 3.1 and 3.3	9.87E-02
HEBBDFM MARE for k = 4 by theorems 3.1 and 3.3	3.53E-03
CSSEMM MARE for k = 2 Evelyn (2000)	4.76E-01
CSSEMM MARE for k = 3 Evelyn (2000)	9.17E-01
CSSEMM MARE for k = 4 Evelyn (2000)	1.62E-01
EMM MARE for k = 2 Bahar (2019)	1.84E+00
EMM MARE for k = 3 Bahar (2019)	2.47E-01
EMM MARE for k = 4 Bahar (2019)	9.73E-01
BSM MARE for k = 2 Osu et al (2021)	7.04E-01
BSM MARE for k = 3 Osu et al (2021)	7.05E-01
BSM MARE for k = 4 Osu et al (2021)	7.06E-01

TABLE 4.2.2: Comparison between the Minimum Absolute Random Errors (MAREs) of HEBBDFM after the incorporations of the evaluated theorem 4.1 with theorems 4.2 and 4.3 for k = 2, 3 and 4 with [9, 10, 18] at constant step size h = 0.01 for Example 2.

Numerical Method	**Compared MAREs with** [9, 10, 18]
HEBBDFM MARE for k = 2 by theorems 3.1 and 3.2	7.46E-04
HEBBDFM MARE for k = 3 by theorems 3.1 and 3.2	1.01E-01
HEBBDFM MARE for k = 4 by theorems 3.1 and 3.2	2.91E-02
HEBBDFM MARE for k = 2 by theorems 3.1 and 3.3	5.22E-01
HEBBDFM MARE for k = 3 by theorems 3.1 and 3.3	1.95E-01
HEBBDFM MARE for k = 4 by theorems 3.1 and 3.3	9.55E-02
CSSEMM MARE for k = 2 Evelyn (2000)	4.76E-01
CSSEMM MARE for k = 3 Evelyn (2000)	9.17E-01
CSSEMM MARE for k = 4 Evelyn (2000)	1.62E-01
EMM MARE for k = 2 Bahar (2019)	1.84E+00
EMM MARE for k = 3 Bahar (2019)	2.47E-01
EMM MARE for k = 4 Bahar (2019)	9.73E-01
BSM MARE for k = 2 Osu et al (2021)	7.04E-01
BSM MARE for k = 3 Osu et al (2021)	7.05E-01
BSM MARE for k = 4 Osu et al (2021)	7.06E-01

4.3 Graphical Presentation for Compared Results

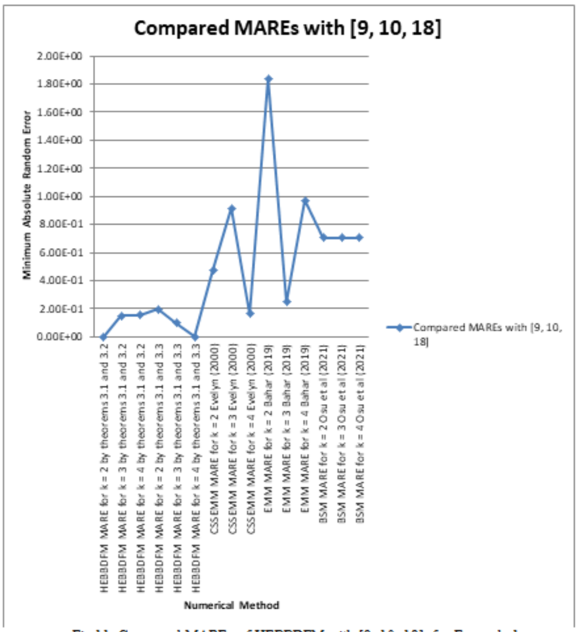

Fig.11: Compared MAREs of HEBBDFM with [9, 10, 18] for Example 1

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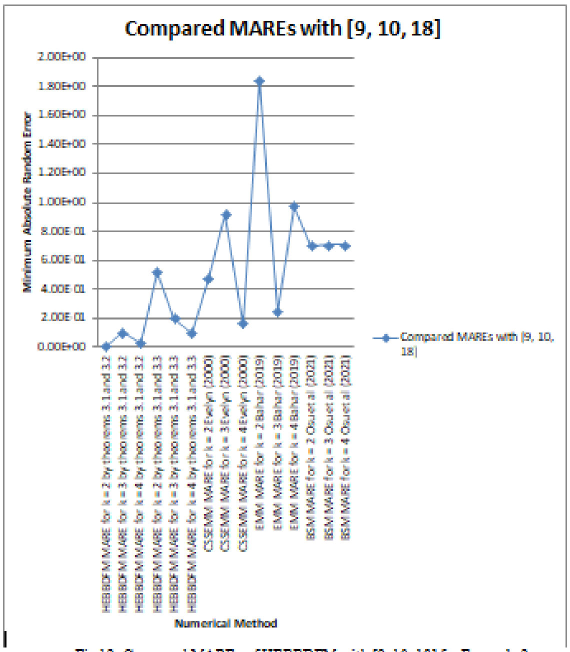

Fig 12: Compared MAREs of HEBBDFM with [9, 10, 18] for Example 2

5. Conclusion

In this study, we have proven and displayed that Hybrid Extended Block Backward Differentiation Formulae Method (HEBBDFM) for step numbers k = 2, 3, and 4 are suitable for approximate solution of Stochastic Delay Differential Equation (SDDE). As observed in tables 4.1 to 4.4 and figures 7 to 10, the numerical results of the discrete schemes of lower step number $k = 2$ of HEBBDFM performed slightly better and faster than the higher step numbers $k = 3$ and 4 by producing the Least Minimum Absolute Random Error (LMARE). In comparing the numerical results of this method with other existing methods in literature as shown in tables 4.2.1 to 4.2.2 and figures 11 to 12, the newly developed mathematical expressions of theorems 3.1 and 3.2 for the evaluations of the delay term and the noise term in solving some SDDEs with the discrete schemes of HEBBDFM gives better results by producing Least Minimum Absolute Random Error (LMARE) in a Lesser Computational Processing Unit Time (LCPUT) faster than the theorems of 3.1 and 3.3 and other existing methods that applied interpolation techniques in evaluations of the delay term and the noise term. Further research should be carried out for step numbers $k = 5, 6, 7, \ldots$ on the approximate solutions of SDDEs using HEBBDFM.

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